Relativistic motion equation solver

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Task Investigate an relativistic motion of charged particle in the presence of electromagnetic field using numerical simulation!

Physics The motion of the particle which has the mass m and charge q in the simultaneous presence of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is described by set of first order differential motion equations with Lorentz force on the right side

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1}$$

where $\mathbf{v}$ is a velocity of the particle,

$$\mathbf{p} = \gamma m \mathbf{v} \tag{2}$$

its momentum and $\gamma = 1/\sqrt{1 - \mathbf{v}^2/c^2}$ is Lorentz factor. This vector equation can be understood as the set of three scalar equations

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = q(E_x + v_y B_z - v_z B_y), \tag{3}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = q(E_y + v_z B_x - v_x B_z), \tag{4}$$

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = q(E_z + v_x B_y - v_y B_x). \tag{5}$$

Since the expression of $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$ is not simple because of the dependence $\gamma = \gamma(\mathbf{v})$, it is useful to introduce energy of particle

$$\mathcal{E} = \sqrt{m^2c^4 + \mathbf{p}^2c^2} = \gamma mc^2. \tag{6}$$

The Lorentz factor γ can be expressed from the equation (6) as

$$\gamma = \frac{\sqrt{m^2c^4 + \mathbf{p}^2c^2}}{mc^2} \tag{7}$$

and combining with the equation (2) the particle velocity can be expressed as

$$\mathbf{v} = \frac{\mathbf{p}c^2}{\sqrt{m^2c^4 + \mathbf{p}^2c^2}}. \tag{8}$$

Implementation The evolution of particle's trajectory and momentum should be investigated. Set of six differential equations will be integrated, particularly

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p_x c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}, \tag{9}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{p_y c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}, \tag{10}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{p_z c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}, \tag{11}$$

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = q\left(E_x + \frac{p_y c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_z - \frac{p_z c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_y\right), \tag{12}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = q\left(E_y + \frac{p_z c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_x - \frac{p_x c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_z\right), \tag{13}$$

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = q\left(E_z + \frac{p_x c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_y - \frac{p_y c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2}}B_x\right). \tag{14}$$

Six initial conditions

$$x(0) = x_0 \qquad y(0) = y_0 \qquad z(0) = z_0$$

$$p_x(0) = p_{x0} \qquad p_y(0) = p_{y0} \qquad p_z(0) = p_{z0}$$

together with the information about time evolution of electromagnetic field

$$\mathbf{E} = \mathbf{E}(t) \qquad \mathbf{B} = \mathbf{B}(t)$$

need to be known for successful integration of those equations.

This set of equations can be understood as a vector equation

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{Y}) \tag{15}$$

$$\mathbf{Y}(0) = \mathbf{Y_0}, \tag{16}$$

where

$$\mathbf{Y} = \begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ p_z \end{pmatrix}.$$

This vector equation can be solved e. g. using single step 4th order Runge-Kutta scheme

$$\begin{aligned}
\mathbf{k_1} &= \mathbf{f}(t_n, \mathbf{Y}_n) && (17)\\
\mathbf{k_2} &= \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{Y}_n + \frac{h}{2}\mathbf{k_1}\right) && (18)\\
\mathbf{k_3} &= \mathbf{f}\left(t_n + \frac{h}{2}, \mathbf{Y}_n + \frac{h}{2}\mathbf{k_2}\right) && (19)\\
\mathbf{k_4} &= \mathbf{f}\left(t_n + h, \mathbf{Y}_n + h\mathbf{k_3}\right) && (20)\\
\mathbf{Y}_{n+1} &= \mathbf{Y}_n + \frac{h}{6}\left(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4}\right). && (21)
\end{aligned}$$

Reduction into 2D case Let us consider 2D case, when $B_x = 0$, $B_y = 0$ and $E_z = 0$. When particle's momentum in z direction is equal to zero as well, particle stay in in the same plane whole time. Hence, set of only four ordinary differential equations need to be integrated, in particular

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{p_x c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2)c^2}}, && (22)\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{p_y c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2)c^2}}, && (23)\\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= q\left(E_x + \frac{p_y c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2)c^2}}B_z\right), && (24)\\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= q\left(E_y - \frac{p_x c^2}{\sqrt{m^2c^4 + (p_x^2 + p_y^2)c^2}}B_z\right). && (25)
\end{aligned}$$

Examples of interesting configurations

- Static field, 2D case, E_x = -200 V/m, E_y = 200 V/m, B_z = 0.09 T, $x_0 = 0$, $y_0 = 0$, $p_x = 10^{-24}$ kg·m·s^{-1}, $p_y = 0$.
- Same configuration with oscillating magnetic field, 2D case, $B_z(t) = \frac{B_{z0}}{2}\sin\left(\frac{1.2t\pi}{T}\right)$, $B_{z0} = 0.09$ T, $T = 600$ ps.
- $B_z(t) = \frac{B_{z0}}{2}\sin\left(\frac{2.5t\pi}{T}\right)$
- $B_z(t) = \frac{B_{z0}}{2}\sin\left(\frac{14.5t\pi}{T}\right)$
- Figure out something interesting yourself and tell me!

Bonus task Implement some multistep method!