29. Diffraction of waves

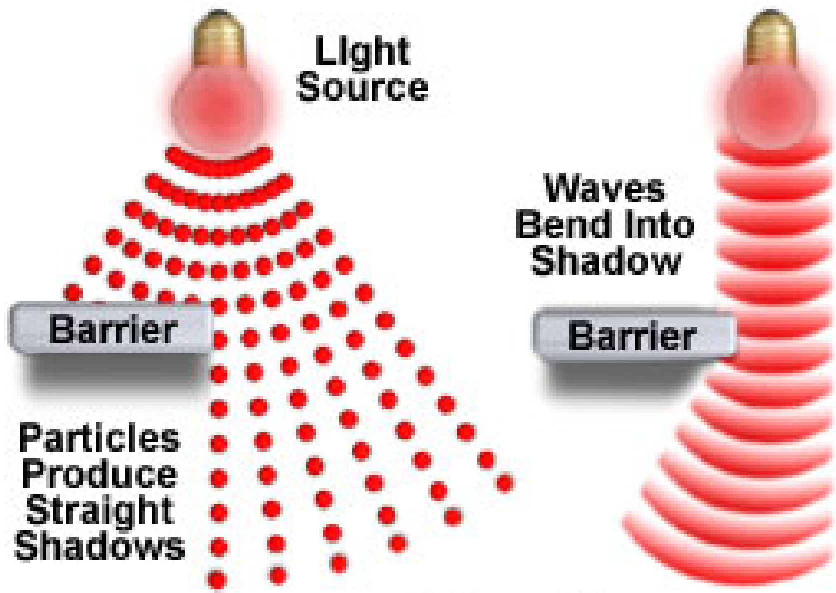
Light bends!

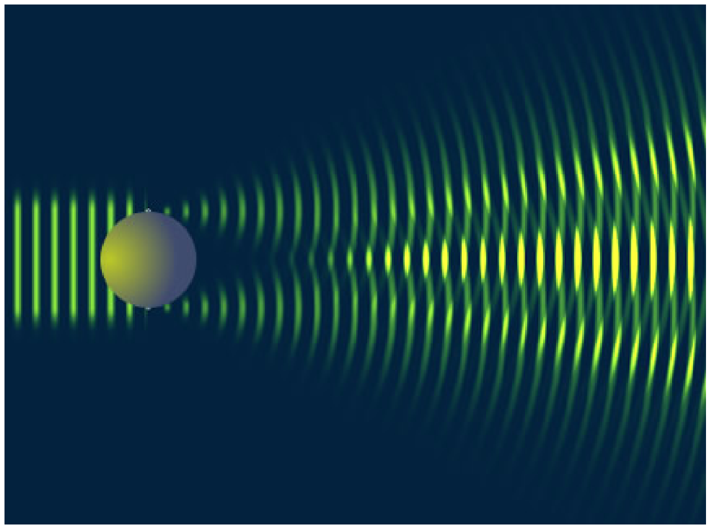
Diffraction assumptions

The Kirchhoff diffraction integral

Fresnel Diffraction

diffraction from a slit





Diffraction

Light does not always travel in a straight line.

It tends to bend around objects. This tendency is called "diffraction.“

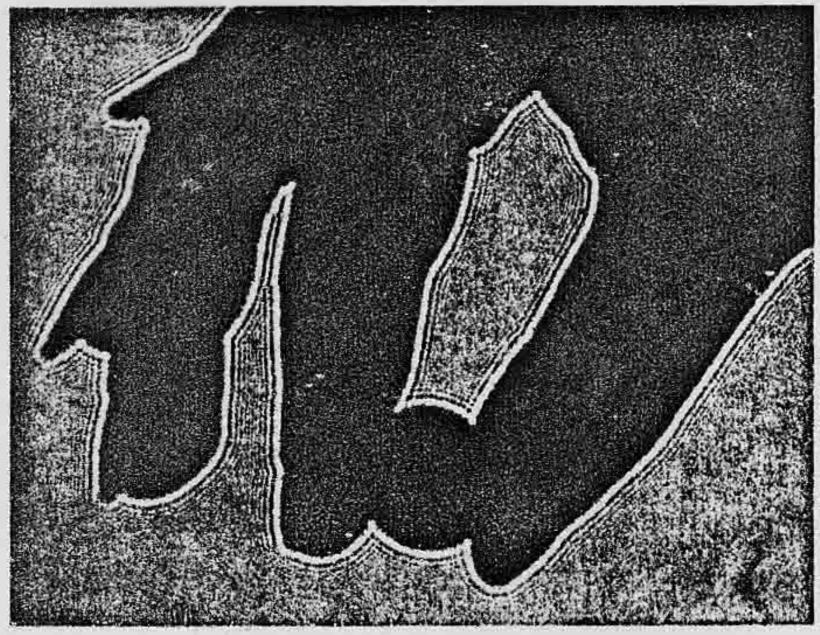
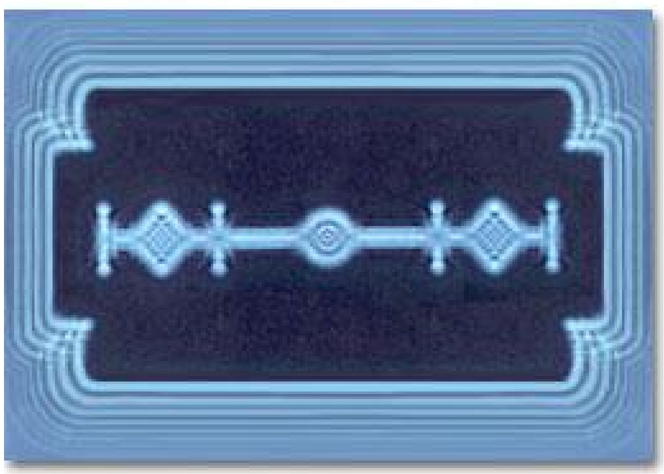
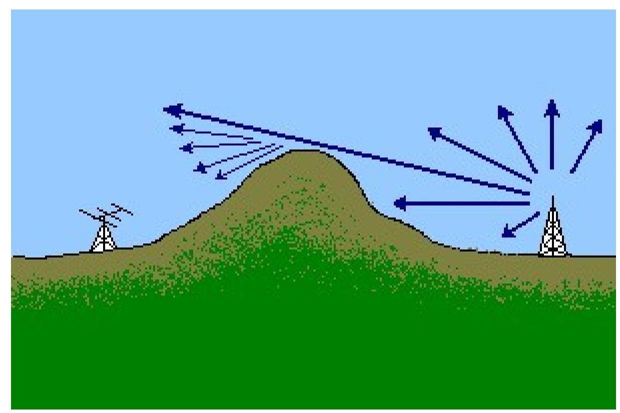


Fig. 10.1 The shadow of a hand holding a dime, cast directly on 4 × 5 Polaroid A.S.A. 3000 film using a He–Ne beam and no lenses.



Shadow of a razor blade illuminated by a laser

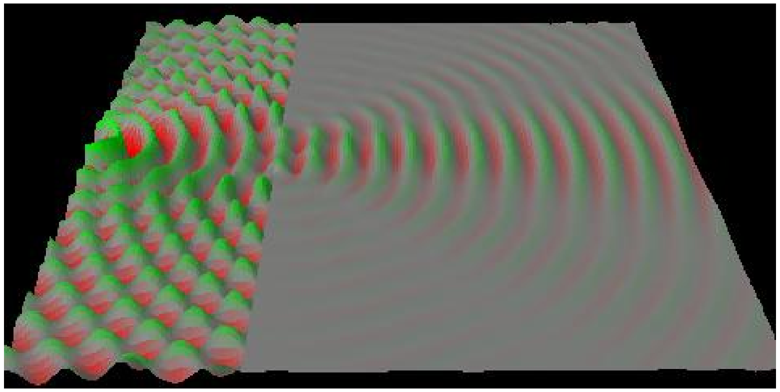


This means that radio communications does not necessarily require a line of sight.

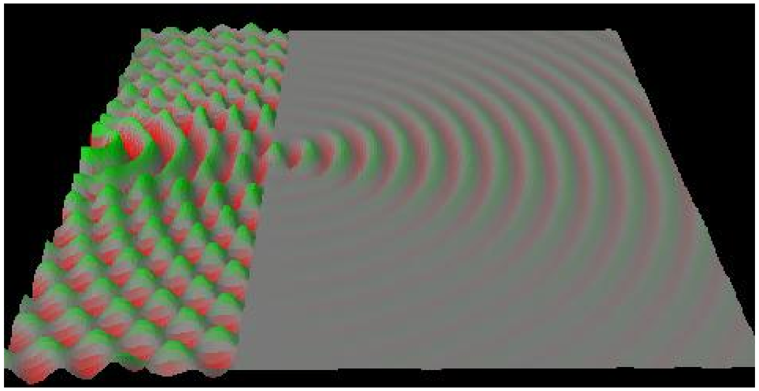
Diffraction of a wave by a slit

Passing light through a small slit yields a diffraction pattern that depends on the size of the slit and the wavelength of the wave.

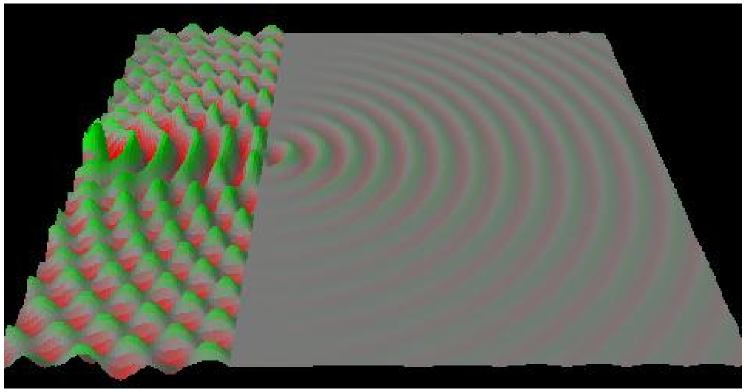
This phenomenon is general, and can be observed using waves of any kind.



Large slit



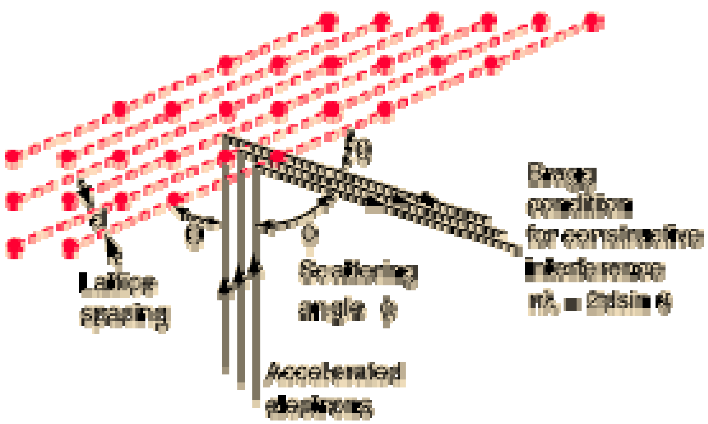
Smaller slit

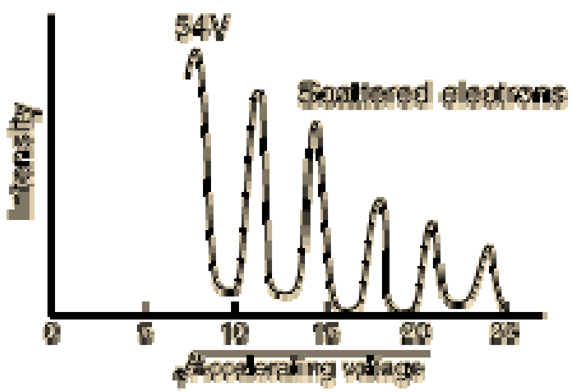


Very small slit

Diffraction of particles

Electrons do this too. The observation of this fact was one of the first important confirmations of quantum physics.



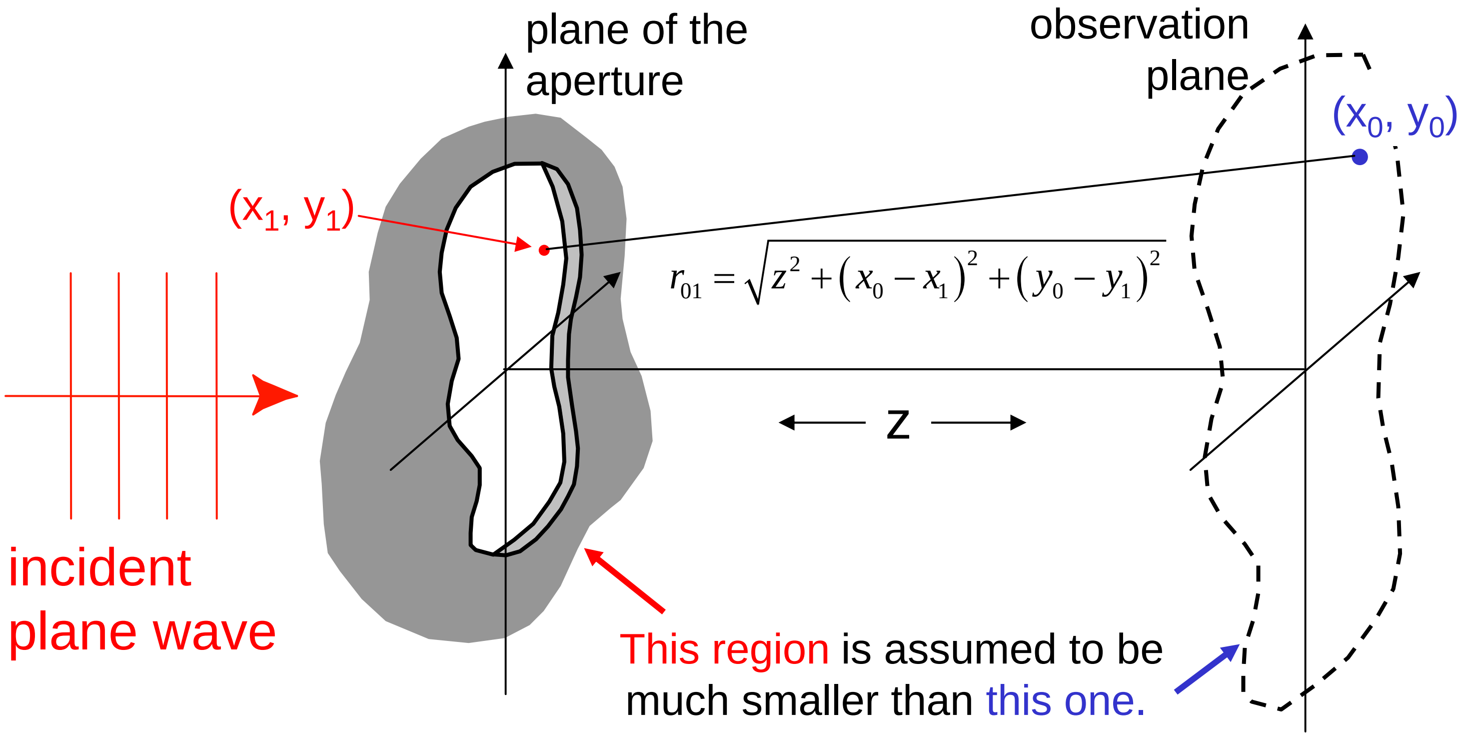


C. J. Davisson and
L. H. Germer, 1927

This experiment was only a few years after Louis de Broglie had described the wavelength of a particle in terms of its momentum, $\lambda = h/p$.
(h = Planck's constant)

The Diffraction Problem

We wish to find the light electric field after a screen with a hole in it. This is a very general problem with far-reaching implications.



What is $E(x_0,y_0)$ at a distance z from the plane of the aperture?

Diffraction Assumptions

Gustav R. Kirchhoff
(1824 - 1887)

The first thorough treatment of this problem was due to Kirchhoff. He made a few assumptions:

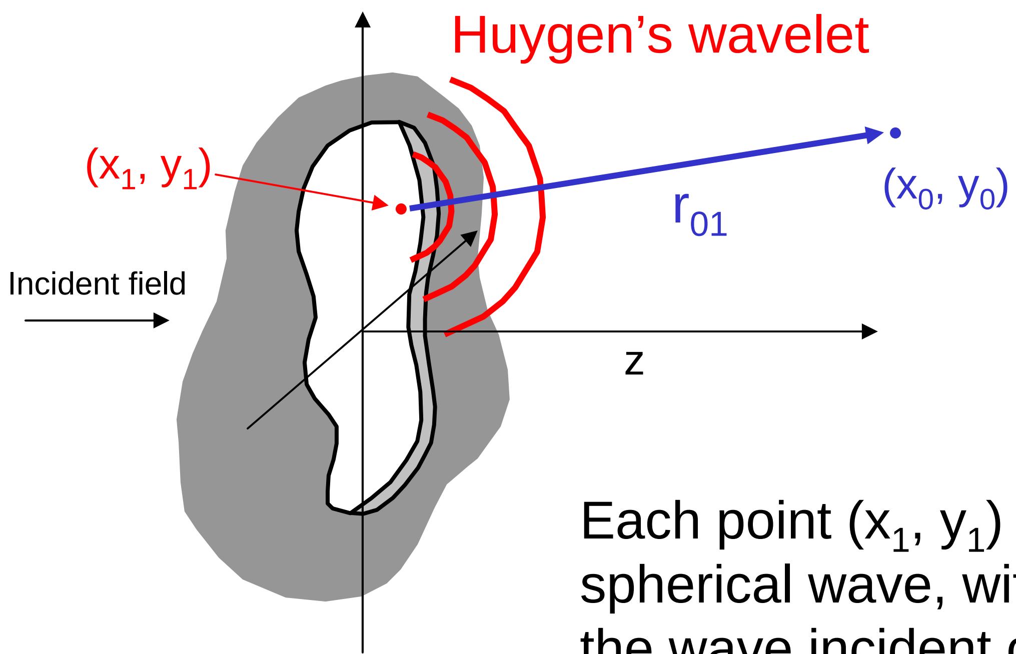
1) Maxwell's equations

2) Inside the aperture, the field and its spatial derivative are the same as if the screen were not present.

3) Outside the aperture (in the shadow of the screen), the field and its spatial derivative are zero.

This set of assumptions actually over-determines the problem, and it can be shown that the diffracted field is zero everywhere. But even so, this is a useful starting point. A more accurate treatment is very complicated!

A solution based on Huygen's principle



$$r_{01} = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}$$

Each point (x_1, y_1) in the aperture emits a spherical wave, with amplitude determined by the wave incident on the aperture.

The net field at the point (x_0, y_0) is therefore given by a superposition:

$$E(x_0, y_0) = \sum_{\substack{\text{all points } (x_1, y_1) \\ \text{in the aperture}}} \left(\text{spherical wave propagating a distance } r_{01}\right) \times \left(\text{incident field at } x_1, y_1\right)$$

Of course, the sum becomes an integral…

The Solution: Kirchhoff Diffraction Integral

The field in the observation plane, $E(x_0,y_0)$, at a distance z from the aperture plane is given by a convolution:

$$E(x_0,y_0) = \iint\limits_{\text{Aperture}(x_1,y_1)} h(x_0 - x_1, y_0 - y_1) E(x_1,y_1) dx_1 dy_1$$

where : $$h(x_0 - x_1, y_0 - y_1) = \frac{1}{j\lambda} \frac{\exp(jkr_{01})}{r_{01}}$$

and : $$r_{01} = \sqrt{z^2 + \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2}$$

A very complicated result! In order to use this, we must make some approximations…

Paraxial approximation

First, we note that we can factor z out of the square root in the expresson for r_{01}:

$$r_{01} = z\sqrt{1+\left(\frac{x_0-x_1}{z}\right)^2+\left(\frac{y_0-y_1}{z}\right)^2}$$

In the spirit of the paraxial approximation, we will assume that the aperture is small compared to the distance z, so that $z >> x_0 - x_1$ and $y_0 - y_1$.

Make use of the Taylor expansion: $\sqrt{1+\varepsilon} \approx 1+\frac{1}{2}\varepsilon$

$$r_{01} \approx z\left[1+\frac{1}{2}\left(\frac{x_0-x_1}{z}\right)^2+\frac{1}{2}\left(\frac{y_0-y_1}{z}\right)^2\right] = z+\frac{(x_0-x_1)^2}{2z}+\frac{(y_0-y_1)^2}{2z} = z + \text{small corrections}$$

Replace r_{01} in both the exponent and the denominator of $h(x_0-x_1, y_0-y_1)$:

$$h(x_0-x_1, y_0-y_1) \approx \frac{1}{j\lambda}\frac{\exp(jkz)\exp\left(jk\frac{(x_0-x_1)^2}{2z}\right)\exp\left(jk\frac{(y_0-y_1)^2}{2z}\right)}{z+\frac{(x_0-x_1)^2}{2z}+\frac{(y_0-y_1)^2}{2z}}$$

We will neglect this term.

Paraxial approximation: Fresnel diffraction

Thus, we have:

$$E(x_0, y_0) \approx \iint_{Aperture(x_1,y_1)} \frac{1}{j\lambda z} \exp\left\{ jk \left[z + \frac{\left(x_0 - x_1\right)^2}{2z} + \frac{\left(y_0 - y_1\right)^2}{2z} \right] \right\} E(x_1, y_1) dx_1 dy_1$$

Multiplying out the squares, and factoring out the constants:

$$E\left(x_0, y_0\right) = \frac{e^{jkz}}{j\lambda z} \iint_{Aperture(x_1,y_1)} \exp\left\{ jk \left[\frac{(x_0^2 - 2x_0 x_1 + x_1^2)}{2z} + \frac{(y_0^2 - 2y_0 y_1 + y_1^2)}{2z} \right] \right\} E(x_1, y_1) dx_1 dy_1$$

If the incident wave is a plane wave, as is typically assumed, then:

$$E(x_1, y_1) = \text{constant}$$

(constant with respect to x_1 and y_1)

The Fresnel Diffraction Integral

Usually, instead of writing an integral over an aperture, we will explicitly write the aperture function in the integral:

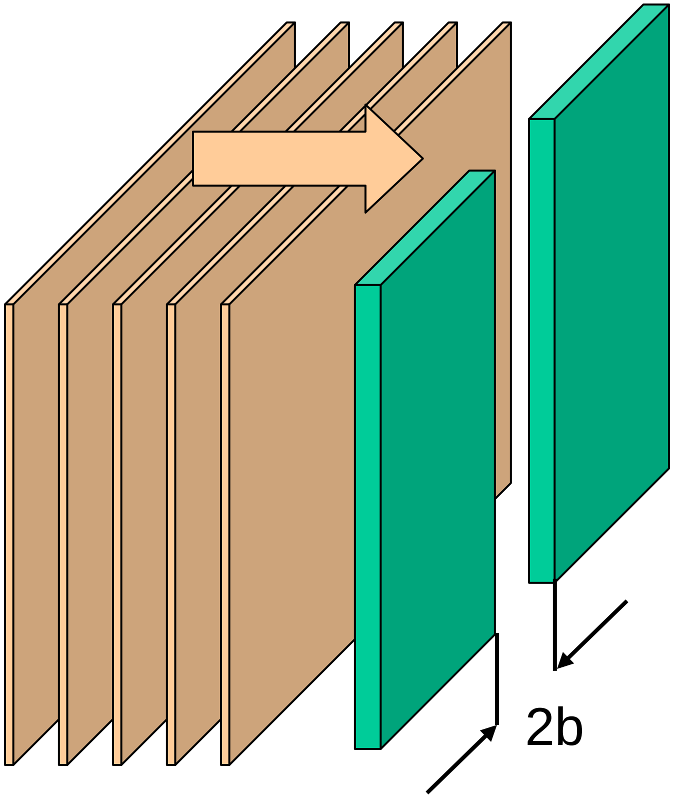
$$E(x_0, y_0) = \frac{E_{in}}{j\lambda z} \exp\left(jk \left(z + \frac{x_0^2 + y_0^2}{z} \right) \right) \iint \exp\left\{ jk \left[\frac{(-2x_0 x_1 - 2y_0 y_1)}{2z} + \frac{(x_1^2 + y_1^2)}{2z} \right] \right\} Aperture(x_1, y_1)\, dx_1 dy_1$$

And we'll usually neglect the factors in front of the integral, to obtain:

$$E(x_0, y_0) \propto \iint \exp\left\{ jk \left[\frac{(-2x_0 x_1 - 2y_0 y_1)}{2z} + \frac{(x_1^2 + y_1^2)}{2z} \right] \right\} Aperture(x_1, y_1)\, dx_1 dy_1$$

This is the Fresnel Diffraction integral. Even with all the approximations we've made, it is usually difficult to evaluate.

Fresnel diffraction example: a slit

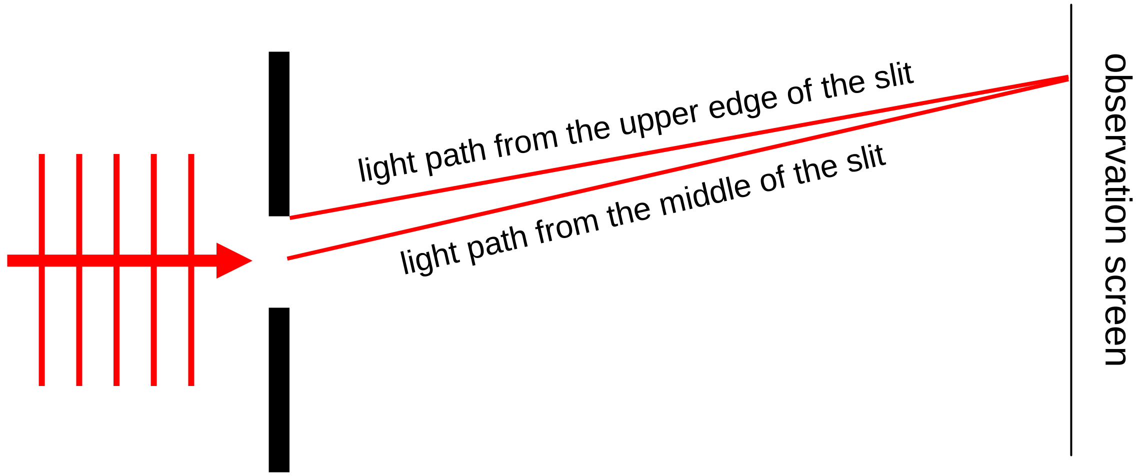


Consider a uniform plane wave incident on a metal screen with a slit of width 2b in the x_1-direction. A one-dimensional problem, this may be the simplest of all possible diffraction problems.

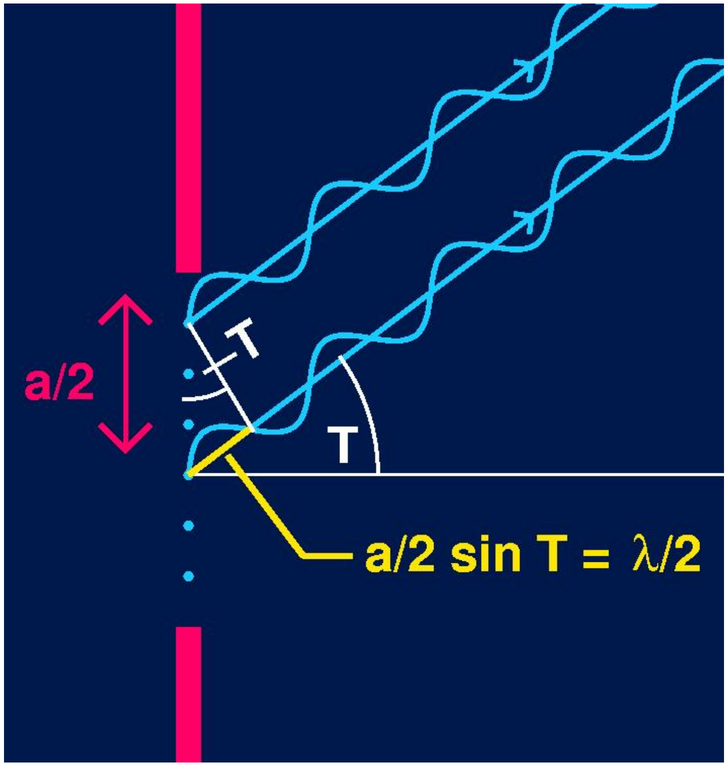
It's still not easy.

Before solving it, let's first try to anticipate what we might expect the answer to look like.

Destructive interference when the path length difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, etc.

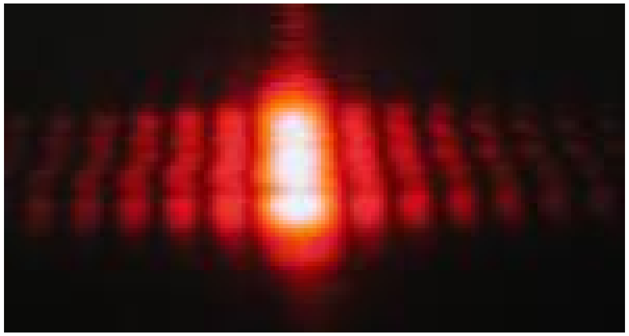


Diffraction causes fringes



At a certain angle **T**, the path difference between two waves (from the top of the slit and the mid-point) equals half of a wavelength. This leads to destructive interference, and therefore a dip in the intensity at that angle.

More generally, we can imagine dividing the slit into an even number of zones. At certain angles, the light from each zone can destructively interfere with the light from the neighboring zone, leading to dark regions in the diffraction pattern.



These alternating light and dark regions are known as “fringes”.

Fresnel integral for a slit

Write the Fresnel integral for this one-dimensional problem:

$$E(x_0) \propto \int \exp\left\{ jk \left[\frac{x_0^2 - 2x_0 x_1 + x_1^2}{2z} \right] \right\} Aperture(x_1)\, dx_1$$

The aperture function is given by: $Aperture(x_1) = \begin{cases} 1 & -b < x_1 < b \\ 0 & \text{otherwise} \end{cases}$

Thus the integral becomes: $E(x_0) = \int_{-b}^{b} \exp\left\{ jk \left[\frac{(x_0 - x_1)^2}{2z} \right] \right\} dx_1$

Next step: define new variables $\xi_1 = \frac{x_1}{b}$ and $\xi_0 = \frac{x_0}{b}$

Then: $E(x_0) \rightarrow E(\xi_0) \propto \int_{-1}^{1} \exp\left\{ \frac{j\pi b^2}{\lambda z} (\xi_0 - \xi_1)^2 \right\} d\xi_1$ since $k = \frac{2\pi}{\lambda}$

Fresnel diffraction example: a slit

$$E(\xi_0) \propto \int_{-1}^{1} \exp\left\{\frac{j\pi b^2}{\lambda z}(\xi_0 - \xi_1)^2\right\} d\xi_1$$

As a shorthand, we define a dimensionless quantity known as the "Fresnel number":

$$N = \frac{b^2}{\lambda z}$$

$$E(\xi_0) \propto \int_{-1}^{1} \exp\left\{j\pi N(\xi_0 - \xi_1)^2\right\} d\xi_1$$

and, of course, we are really interested in the intensity $I(\xi_0) \propto \left|E(\xi_0)\right|^2$

This is not an integral that can be solved in closed form.
It must be computed numerically.

Fresnel Diffraction through a slit: numerical results for $I(\xi_0)$

Fresnel number: $N = \dfrac{b^2}{\lambda z}$

Example: green light (λ = 0.5 μm)

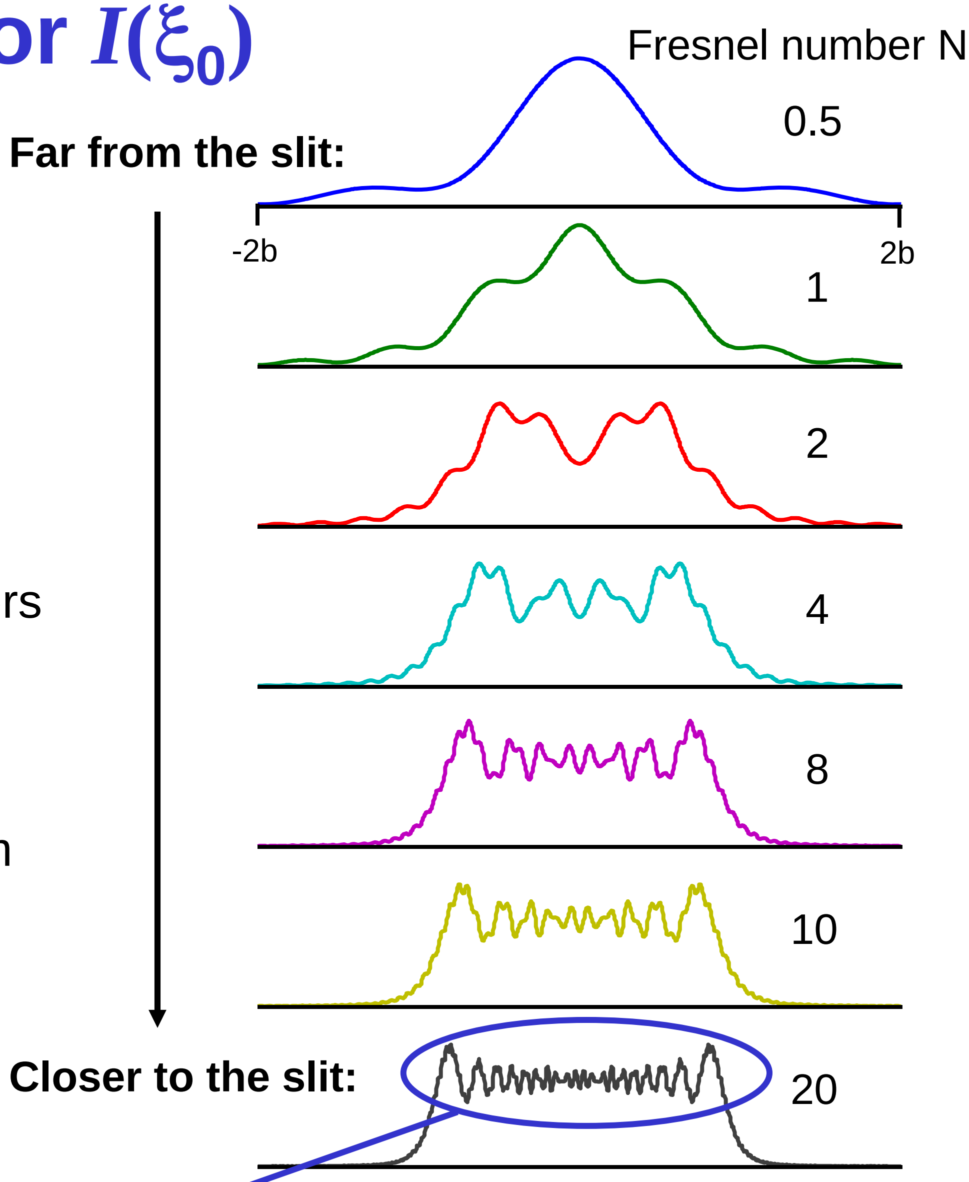
a) slit width b = 1 millimeter = 2000λ

→ N = 1 at a distance of 2 meters

b) slit width b = 10 microns = 20λ

→ N = 1 at a distance of 200 μm

Recall: this Fresnel calculation is only valid for z >> b, which is the paraxial approximation.



# of ripples = Fresnel number!

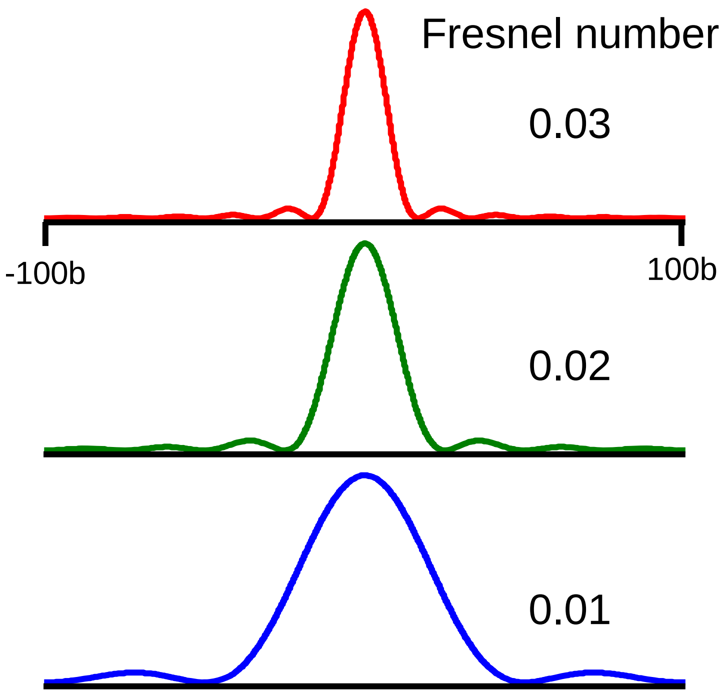
Fresnel Diffraction through a slit: far field

$$E(\xi_0) \propto \int_{-1}^{1} \exp\left\{ j\pi N (\xi_0 - \xi_1)^2 \right\} d\xi_1$$

In the limit that N << 1 (very far from the aperture), the integral can be performed analytically. The math is a bit tedious, so we just quote the result here:

$$E(x_0) \propto \frac{\sin\left(\frac{2\pi N x_0}{b}\right)}{\frac{2\pi N x_0}{b}}$$

Our old friend the sinc function!



In this regime (the "far field"), the diffraction pattern no longer changes shape as z increases, but merely expands in size uniformly.

Fresnel Diffraction through a slit: isn't there an awesome java applet that illustrates all this?

More than one, actually

http://www.falstad.com/ripple/

http://www.falstad.com/diffraction/