19. The Fourier Transform in optics

What is the Fourier Transform?

Anharmonic waves

The spectrum of a light wave

Fourier transform of an exponential

The Dirac delta function

The Fourier transform of $e^{j\omega t}$, $\cos(\omega t)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp\left(j\omega t\right)d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\exp\left(-j\omega t\right)dt$$

There's no such thing as exp($j\omega t$)

All semester long, we've described electromagnetic waves like this:

$$E(t) = \mathrm{Re}\left\{ E_0 e^{j\omega t} \right\}$$

What's wrong with this description? Well, what is its behavior after a long time?

$$\lim_{t \to \infty} E(t) = ???$$

In the real world, signals never last forever. We must always expect that:

$$\lim_{t \to \pm\infty} E(t) = 0$$

This means that no wave can be perfectly "monochromatic". All waves must be a superposition of different frequencies.

Jean Baptiste Joseph Fourier, our hero

Fourier went to Egypt with Napoleon's army in 1798, and was made Governor of Lower Egypt.

Later, he was concerned with the physics of heat and developed the Fourier series and transform to model heat-flow problems.

He is also generally credited with the first discussion of the greenhouse effect as a source of planetary warming.

Joseph Fourier, 1768 - 1830

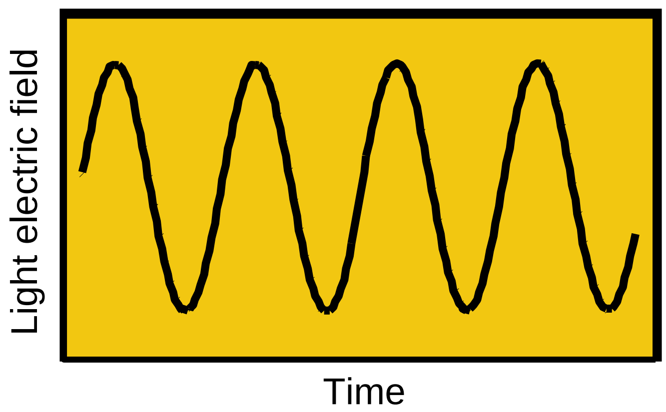
"Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics."

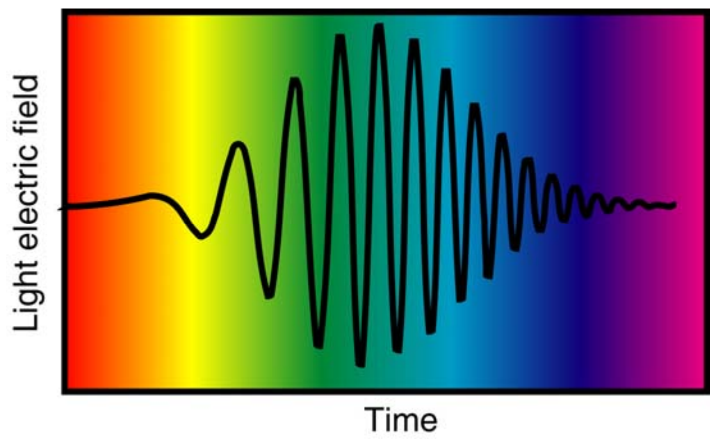
Lord Kelvin

What do we hope to achieve with the Fourier Transform?

We desire a measure of the frequencies present in a wave. This will lead to a definition of the term, the “spectrum.”

Monochromatic waves have only one frequency, ω.





This light wave has many frequencies. And the frequency increases in time (from red to blue).

It will be nice if our measure also tells us when each frequency occurs.

The Fourier Transform and its Inverse

Many of you have seen this in other classes:

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt}$$

Fourier Transform 

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t)\, d\omega}$$

Inverse Fourier Transform 

Be aware: there are different definitions of these transforms. The factor of 2π can occur in several places, but the idea is generally the same.

We often denote the Fourier transform of a function $f(t)$ by $F\{ f(t) \}$, and the inverse transform of a function $g(\omega)$ by $F^{-1}\{ g(\omega) \}$.

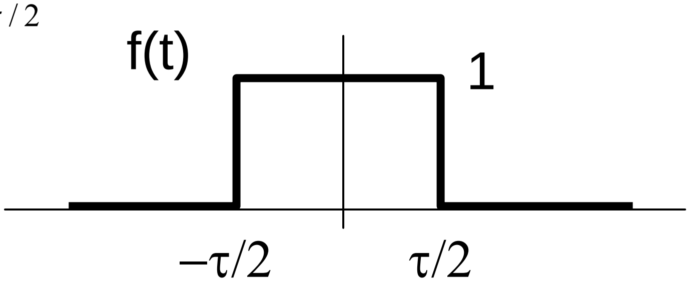
Example: the Fourier Transform of a rectangle function: rect(t)

$$F(\omega) = \int_{-\tau/2}^{\tau/2} \exp(-j\omega t)\, dt = \frac{1}{-j\omega}\left[\exp(-j\omega t)\right]_{-\tau/2}^{\tau/2}$$

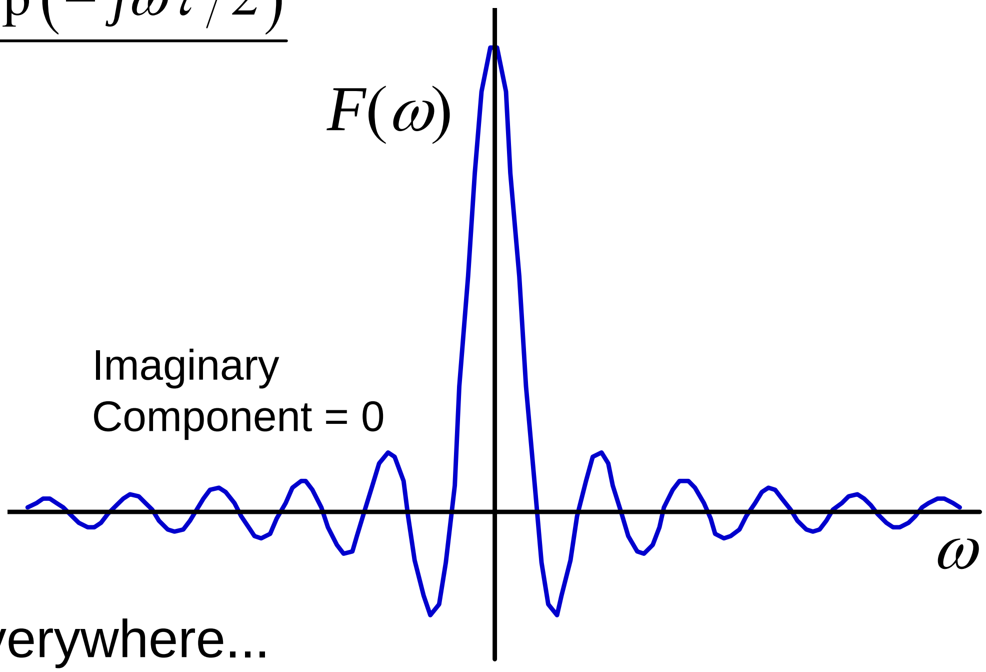
$$= \frac{1}{-j\omega}\left[\exp(-j\omega\tau/2) - \exp(j\omega\tau/2)\right]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(j\omega\tau/2) - \exp(-j\omega\tau/2)}{2j}$$

$$= \tau \cdot \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$



$$\boxed{F(\omega) = \tau \cdot \mathrm{sinc}(\omega\tau/2)}$$



The sinc function crops up everywhere...

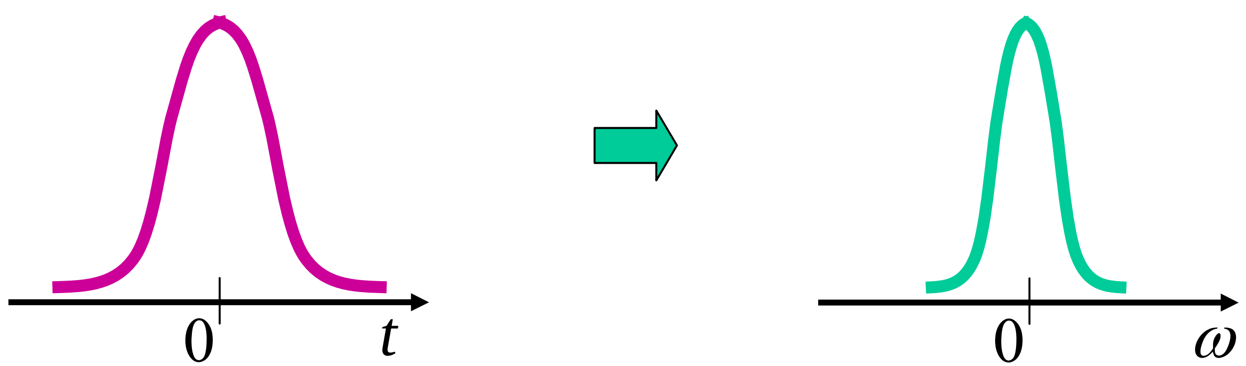
Example: the Fourier Transform of a Gaussian is a Gaussian!

$$F\left\{\exp\left(-at^2\right)\right\} = \int_{-\infty}^{\infty} \exp(-at^2)\exp(-j\omega t)\,dt$$

$$\propto \exp(-\omega^2/4a)$$

There are other examples of functions who are their own Fourier transform.

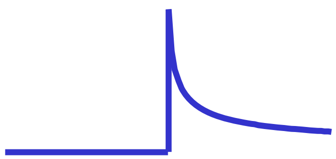
$$\exp(-at^2) \qquad\qquad \exp(-\omega^2/4a)$$



The Fourier transform of 1/sqrt(βt)

Consider the function $t^{-1/2}$, starting at $t = 0$: $H(t) \equiv \begin{cases} 0 \text{ if } t < 0 \\ 1/\sqrt{\beta t} \text{ if } t \geq 0 \end{cases}$

$$F\{H(t)\} = \int_0^\infty \frac{1}{\sqrt{\beta t}} \exp(-j\omega t)\, dt$$



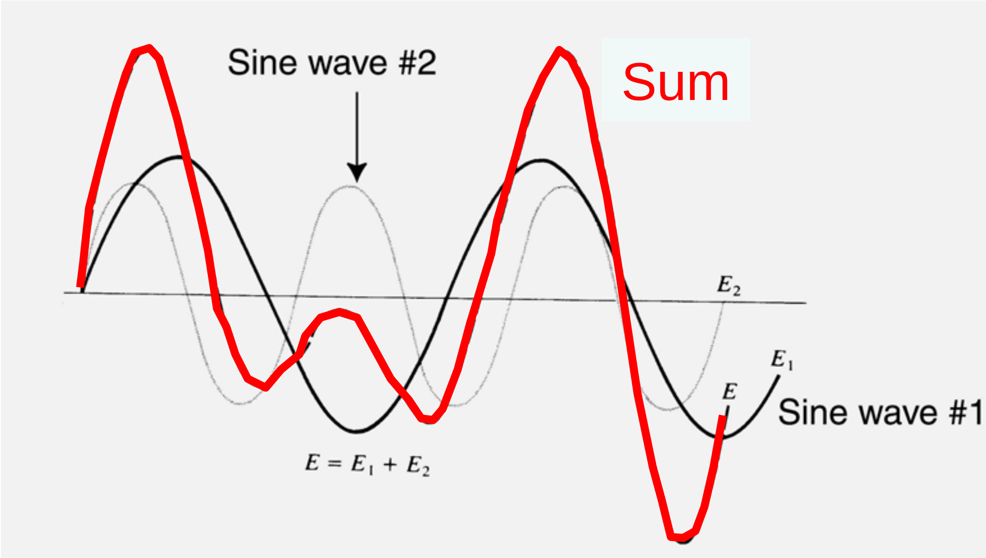
$$= \sqrt{\frac{\pi}{\beta\omega}} e^{-j\pi/4}$$

This is another function which is its own Fourier transform!

This result is significant in the analysis of certain types of random processes - “1/f noise”

Anharmonic waves are sums of sinusoids.

Consider the sum of two sine waves (i.e., harmonic waves) of different frequencies:



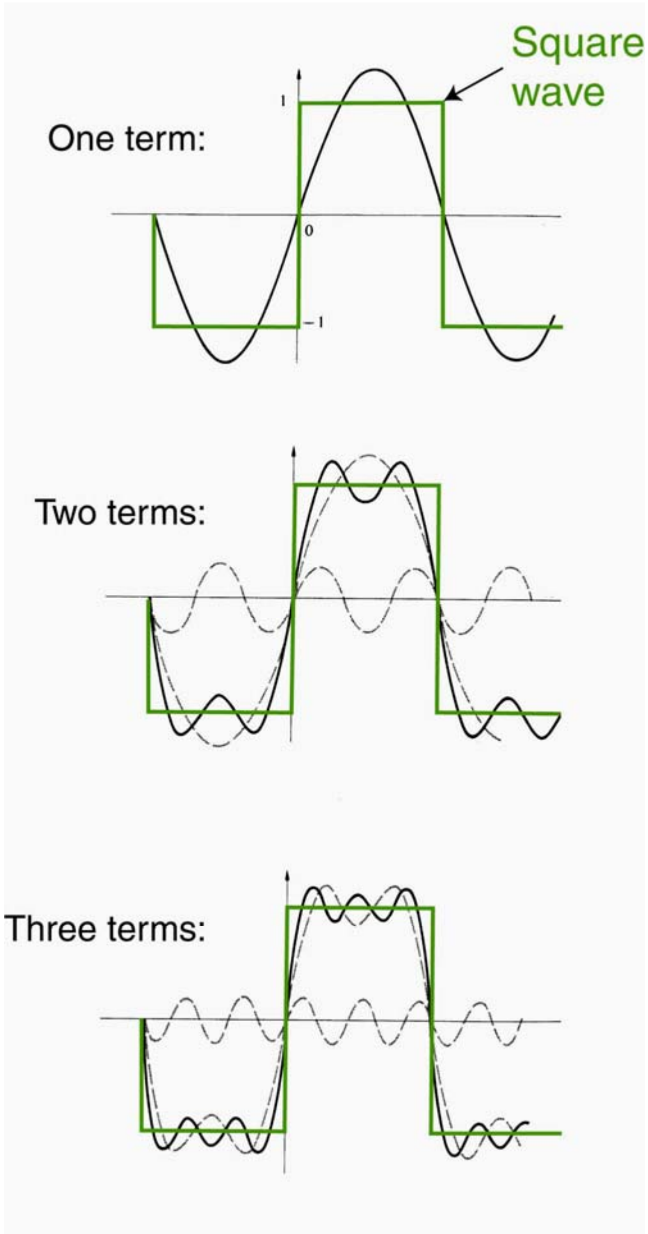
The resulting wave is periodic, but not harmonic.

Essentially all waves are anharmonic. But virtually any anharmonic wave can be written as the sum of harmonic waves.

Fourier analysis is useful because it is the primary tool for handling anharmonic waves.

Fourier decomposing functions

Here, we write a
square wave as
a sum of sine waves.



Some functions don't have Fourier transforms.

A condition for the existence of a given $F(\omega)$ is:

$$\int_{-\infty}^{\infty} |f(t)| \, dt < \infty$$

Functions that do not asymptote to zero in both the $+\infty$ and $-\infty$ directions generally do not have Fourier transforms (with some exceptions).

Of course, such functions don't describe real waves, so this is usually not a problem.

The spectrum of a light wave

We define the spectrum of a wave $E(t)$ to be the magnitude of the square of the Fourier transform:

$$\boxed{S(\omega) = \left| F\{E(t)\} \right|^2}$$

This is our measure of the frequency content of a light wave.

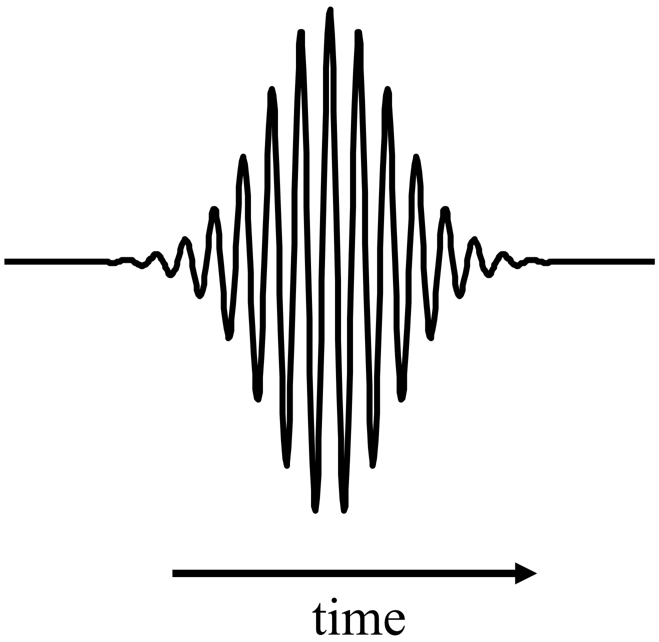
Note that the Fourier transform of $E(t)$ is usually a complex quantity:

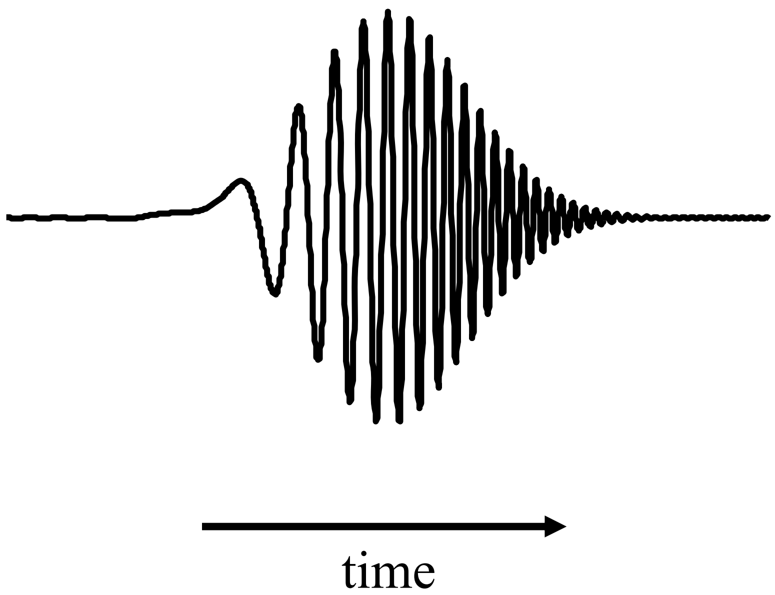
$$F\{E(t)\} = \tilde{E}(\omega) = \left|\tilde{E}(\omega)\right| e^{j\Phi(\omega)}$$

By taking the magnitude, we are throwing away the phase information.
So the spectrum does not contain all of the information about the wave.
It does not contain the *spectral phase*, which is sometimes important.

The spectrum of a light wave does not contain all of the information about E(t)

These two E-fields E(t) have the same spectrum.
But they are obviously different from each other.





The difference is contained in the spectral phase $\Phi(\omega)$.

Measuring the spectrum of a light wave

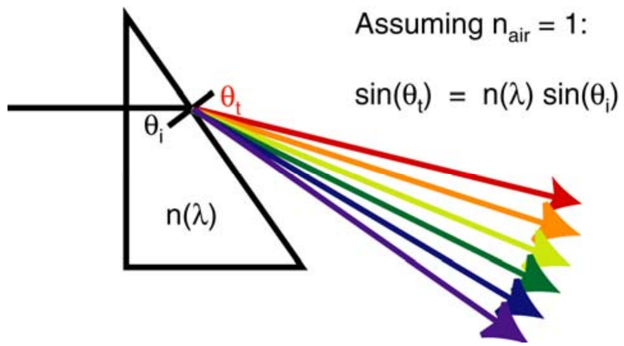
One reason that the spectrum is such an important quantity is that it is easy to measure.

“spectrometer” - a device which measures the spectrum of a signal

If we divide a light wave into its components, and measure the irradiance at each frequency (or equivalently, at each wavelength), then we have measured the spectrum of the light wave.

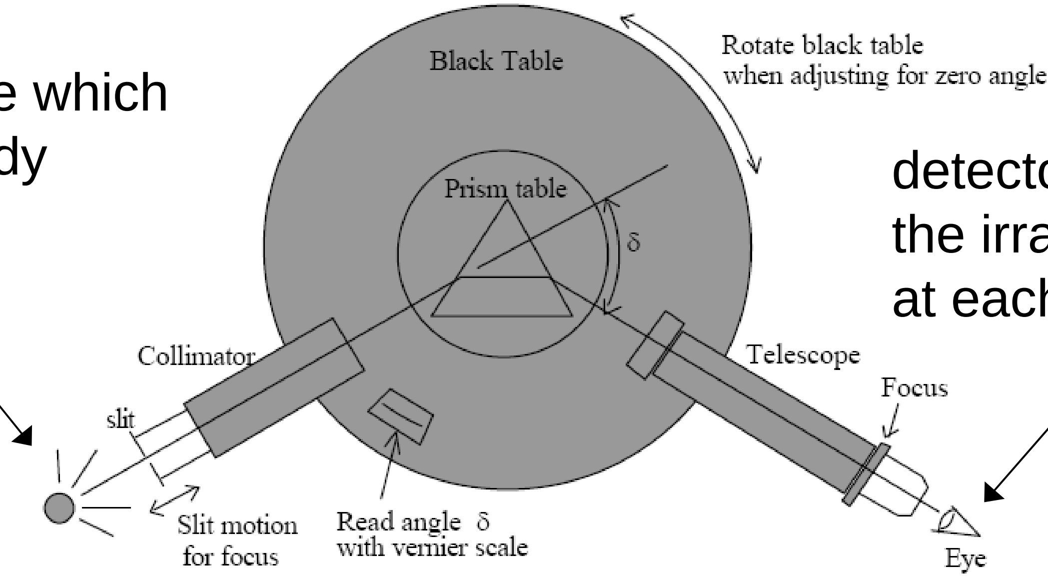
So, how do we divide a light wave into its component parts?

prisms disperse light!

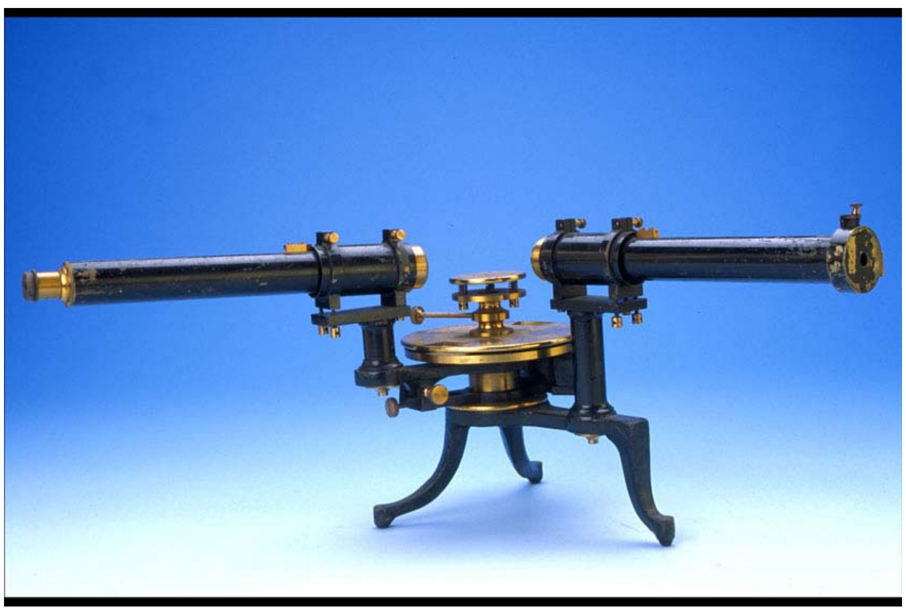


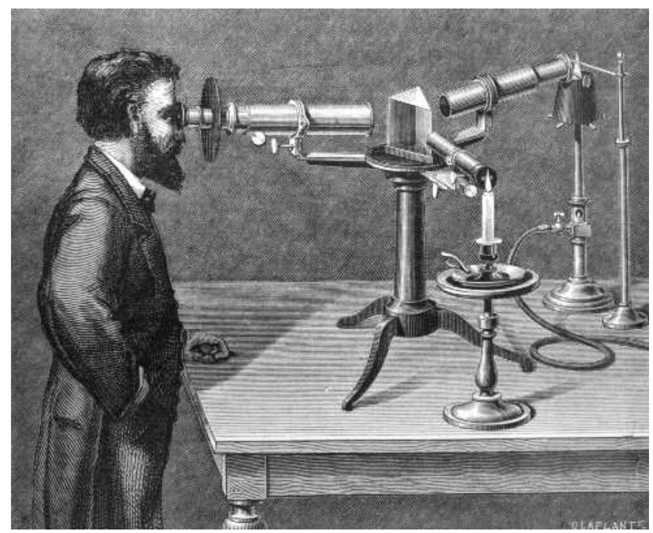
A prism spectrometer

the light source which
we wish to study



detector to measure
the irradiance of light
at each wavelength.

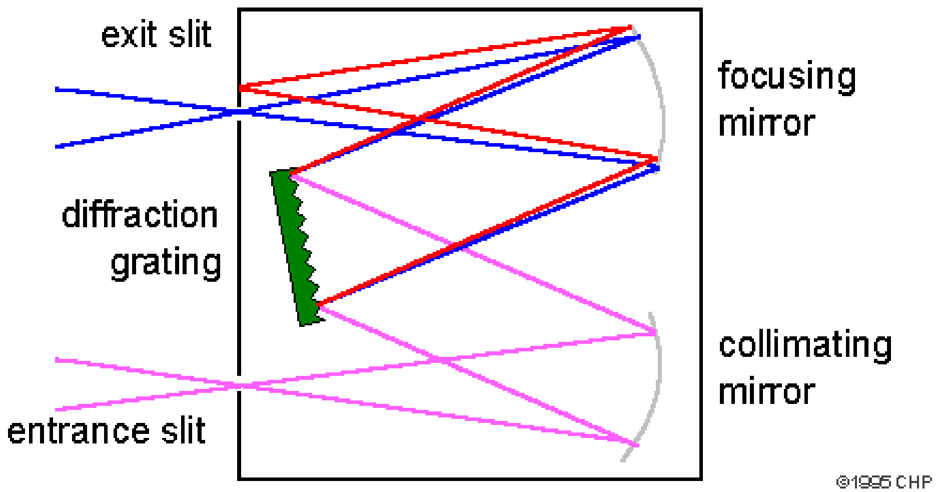


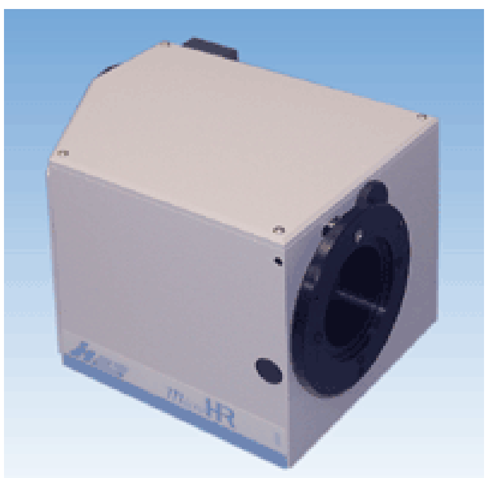


Robert Bunsen, 1859

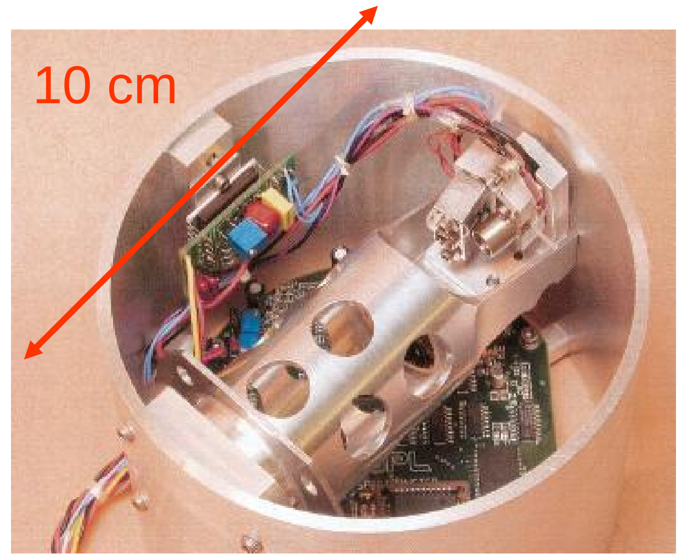
More modern spectrometers

Czerny spectrometer





a miniaturized spectrometer designed by NASA

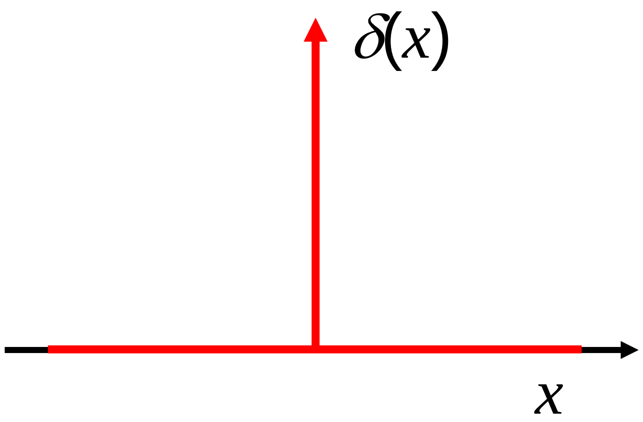


This spectrometer is currently on Mars.

The Dirac delta function - a key tool in Fourier analysis

The Dirac delta function is not really a function at all,
but it is nevertheless very useful.

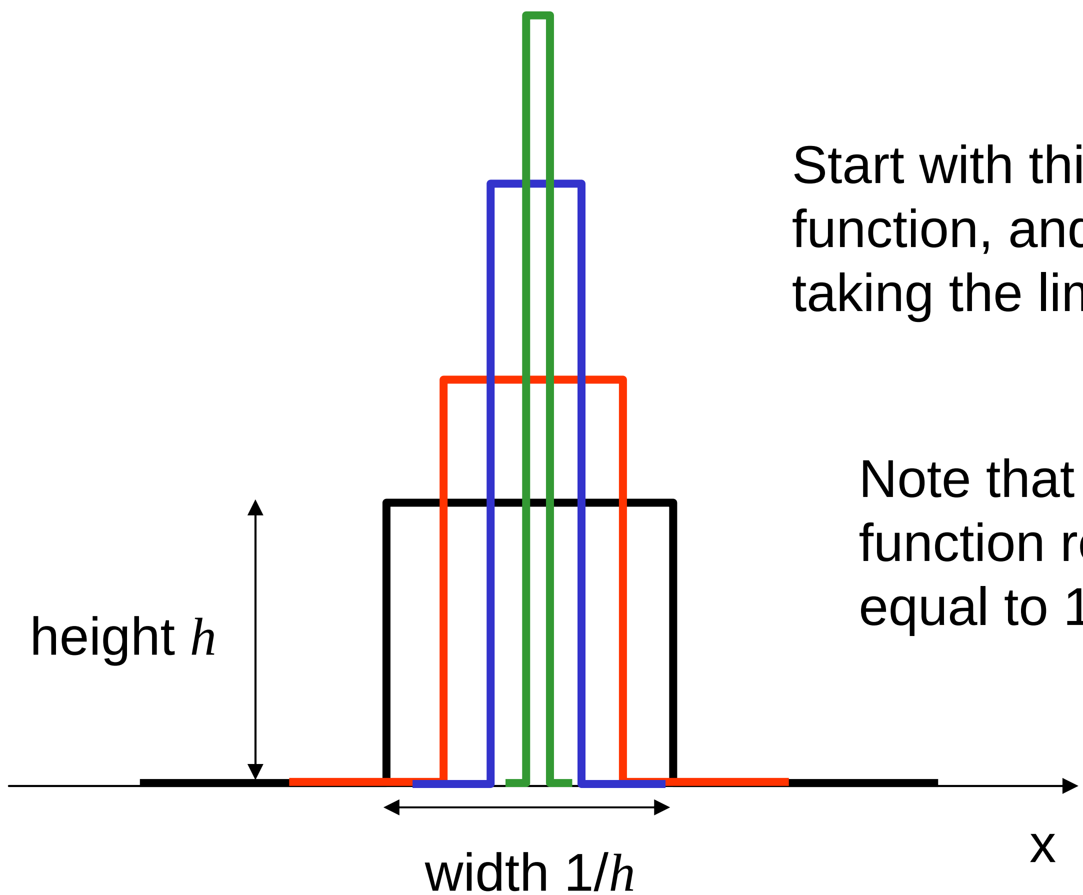
$$\delta(x) \equiv \begin{cases} \infty \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$$



The Dirac delta function

$$\delta(x) \equiv \begin{cases} \infty \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$$

It's best to think of the delta function as a limit of a series of peaked continuous functions.



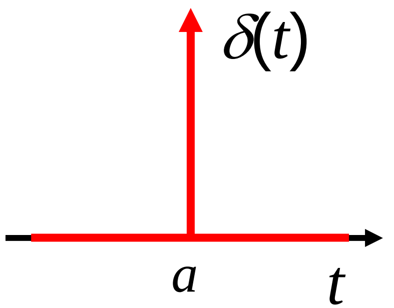
Start with this square function, and imagine taking the limit as $h \to \infty$

Note that the area under the function remains constant, equal to 1.

Dirac δ–function Properties

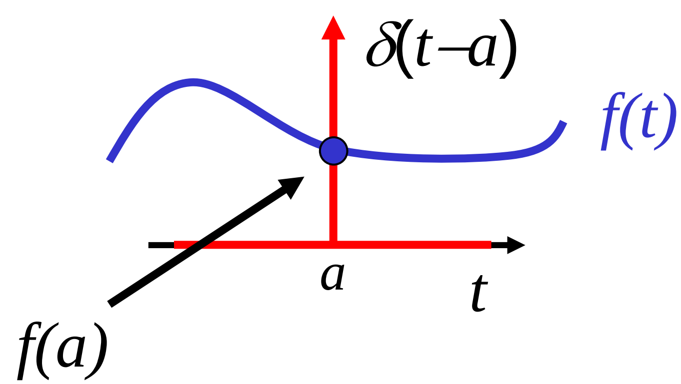
The area under a δ function is one:

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$



Integrating any function f(t) multiplied by a δ function picks out the value of f(t) at the location of the δ function:

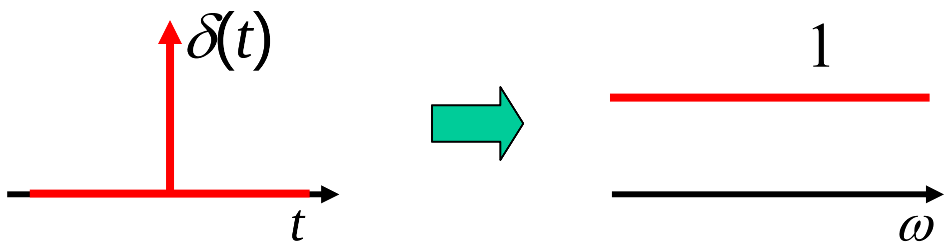
$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a)$$



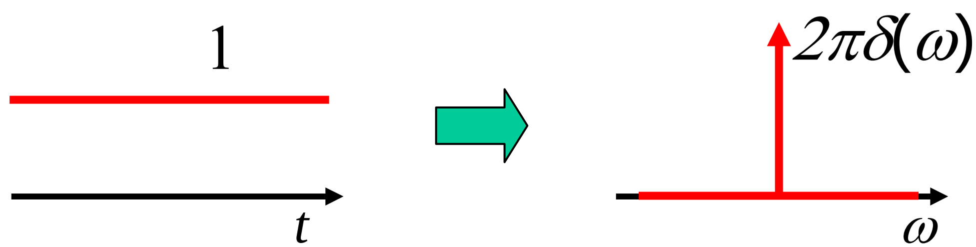
Fourier Transforms and $\delta(t)$

The Fourier transform of $\delta(t)$ is one.

$$\int_{-\infty}^{\infty} \delta(t) \exp(-j\omega t)\, dt = \exp(-j\omega[0]) = 1$$



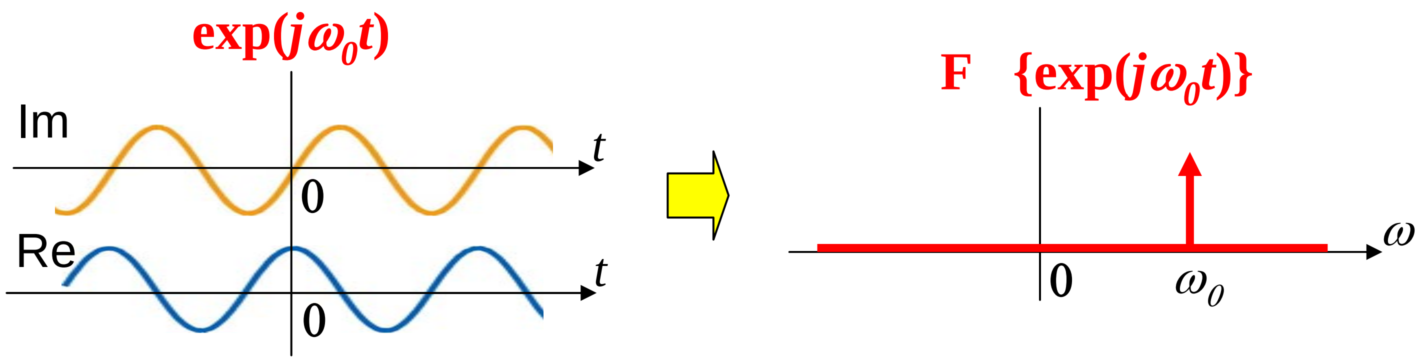
And the Fourier Transform of 1 is $2\pi\delta(t)$: $\int_{-\infty}^{\infty} 1 \exp(-j\omega t)\, dt = 2\pi\, \delta(\omega)$



The Fourier transform of $\exp(j\omega_0 t)$

$$F\{\exp(j\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(j\omega_0 t)\exp(-j\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-j[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$



The function $\exp(j\omega_0 t)$ is the essential component of Fourier analysis. It is a pure frequency - its spectrum consists of only one frequency component.

The Fourier transform of $\cos(\omega_0 t)$

$$F\left\{\cos(\omega_0 t)\right\} = \int_{-\infty}^{\infty} \cos(\omega_0 t)\exp(-j\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(j\,\omega_0\,t) + \exp(-j\,\omega_0\,t)\right]\exp(-j\,\omega t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-j[\omega-\omega_0]t)\,dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-j[\omega+\omega_0]t)\,dt$$

$$= \pi\,\delta(\omega-\omega_0) \quad + \quad \pi\,\delta(\omega+\omega_0)$$