

# Lorentz Oscillator Model

by Dr. Colton, Physics 442/471 (last updated: Winter 2020)

## What is the Lorentz oscillator model?

The Lorentz oscillator model, also known as the Drude-Lorentz oscillator model, involves modeling an electron as a driven damped harmonic oscillator. In this model the electron is connected to the nucleus via a hypothetical spring with spring constant  $C$ . The driving force is the oscillating electric field. The source of the damping force is not specified, but is present so that the oscillations don't go infinite when the driving force is at the resonant frequency. The goal of this model is to (a) use Newton's Second Law to obtain the motion of the electron, which (b) can then be used to obtain expressions for the dipole moment, polarization, susceptibility, and dielectric constant.

## Derivation of electron motion

Let's say the driving oscillating electric field is  $E = E_0 \cos(-\omega t)$ . I'm using  $\cos(-\omega t)$  here instead of  $\cos(+\omega t)$  so that it matches time dependence of a standard traveling EM wave, namely  $\cos(kx - \omega t)$ . The damping force is velocity-dependent, described by damping coefficient  $\gamma$  (units of  $\gamma$  chosen such that force =  $\gamma mv$ ).

$$\begin{aligned} F_{net} &= m\ddot{x} \\ F_{driving} + F_{spring} + F_{damping} &= m\ddot{x} \\ qE_0 \cos(-\omega t) - Cx - \gamma m\dot{x} - m\ddot{x} \\ \ddot{x} + \gamma\dot{x} + \frac{C}{m}x &= \frac{qE_0}{m} \cos(-\omega t) \quad \rightarrow \text{Let } \omega_0 = \sqrt{C/m} \\ \ddot{x} + \gamma\dot{x} + \omega_0^2 x &= \frac{qE_0}{m} \cos(-\omega t) \end{aligned}$$

That is the equation of motion we need to solve. To solve it we'll use the physicist's favorite trick, which is to guess the form of the answer and plug it into the equation.

Guess  $x = x_0 \cos(-\omega t + \phi)$  as solution  $\rightarrow \tilde{x} = x_0 e^{i\phi} e^{-i\omega t}$  in complex notation  
 $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$  (the phase  $\phi$  is lumped in with complex  $\tilde{x}_0$ )

$$\begin{aligned} \text{Derivatives: } \tilde{\dot{x}} &= (-i\omega)\tilde{x}_0 e^{-i\omega t} \\ \tilde{\ddot{x}} &= (-i\omega)^2 \tilde{x}_0 e^{-i\omega t} \end{aligned}$$

Plug into the boxed equation; also convert cosine into a complex exponential:

$$(-i\omega)^2 \tilde{x}_0 e^{-i\omega t} + \gamma(-i\omega)\tilde{x}_0 e^{-i\omega t} + \omega_0^2 \tilde{x}_0 e^{-i\omega t} = \frac{qE_0}{m} e^{-i\omega t}$$

Cancel the  $e^{-i\omega t}$  factors:

$$\tilde{x}_0(-\omega^2 - i\omega\gamma + \omega_0^2) = \frac{qE_0}{m}$$

$$\tilde{x}_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Complex amplitude of electron's motion

The fact that it is complex just means there is a time delay (phase shift) between the driving electric field and the response of the electron's motion. With the explicit time dependence added back in, we have:

$$\tilde{x} = \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t}$$

To perhaps understand it better, we can convert to polar form. Need to first get real & imaginary parts.

Work with the complex term,  $\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$ :

$$\begin{aligned} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} &\times \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\omega_0^2 - \omega^2 + i\omega\gamma} = \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \\ &= \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} + i \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \quad (\text{now in real + imaginary form}) \end{aligned}$$

$$\text{Real part of } \tilde{x}_0 = \frac{qE_0}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

$$\text{Imaginary part of } \tilde{x}_0 = \frac{qE_0}{m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

Polar form, magnitude and phase:  $\sqrt{\text{Re}^2 + \text{Im}^2} \angle \phi$

Since the real (actual) oscillation is given by the real part of  $\tilde{x}_0$ , this means that:

$$x_{\text{actual}}(t) = \frac{qE_0}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \cos(-\omega t + \phi)$$

## Dipole moment

Since we know we can convert to real form anytime we want, for simplicity let's go back to writing the amplitude in complex form. The complex dipole moment induced by an electron moving like this in an atom (with the nucleus at the origin, stationary so it doesn't contribute to the dipole moment) is given by:

$$\begin{aligned} \mathbf{p} &= \sum q_i \mathbf{r}_i \\ \tilde{\mathbf{p}} &= (q) \left( \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t} \right) \hat{\mathbf{x}} \\ \tilde{p} &= \frac{q^2 E_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t} \quad \text{Complex dipole moment} \end{aligned}$$

## Polarization

The polarization  $\mathbf{P}$  is the dipole moment per volume. If we assume that there are  $N$  electrons per volume (sometimes the symbol  $n$  is used for this), each acting with the same dipole moment as given above, then the complex polarization is given by multiplying the last equation by  $N$ :

$$\begin{aligned}\tilde{p} &= \frac{Nq^2 E_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i\omega t} \\ \tilde{P} &= \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} (E_0 e^{-i\omega t}) \\ \tilde{P} &= \frac{Nq^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tilde{E}\end{aligned}\quad \text{Complex polarization}$$

The complex nature of the proportionality constant means that there's a phase shift between  $\mathbf{P}$  and  $\mathbf{E}$ .

### Susceptibility and Permittivity

Because the susceptibility is defined by  $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$ , we can just read off the susceptibility from that last equation as being the stuff multiplying  $\tilde{E}$ , divided by  $\epsilon_0$ :

$$\tilde{\chi} = \frac{Nq^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

The quantity  $\frac{Nq^2}{m\epsilon_0}$  has units of frequency squared, and its square root is called the *plasma frequency* because it happens to also be the frequency a plasma will naturally oscillate at if the positive and negative charges in the plasma are offset from each other.

$$\omega_p = \sqrt{\frac{Nq^2}{m\epsilon_0}} \quad \text{Defn of plasma frequency}$$

That allows us to write the susceptibility in a nice compact form,

$$\tilde{\chi} = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \text{Complex susceptibility}$$

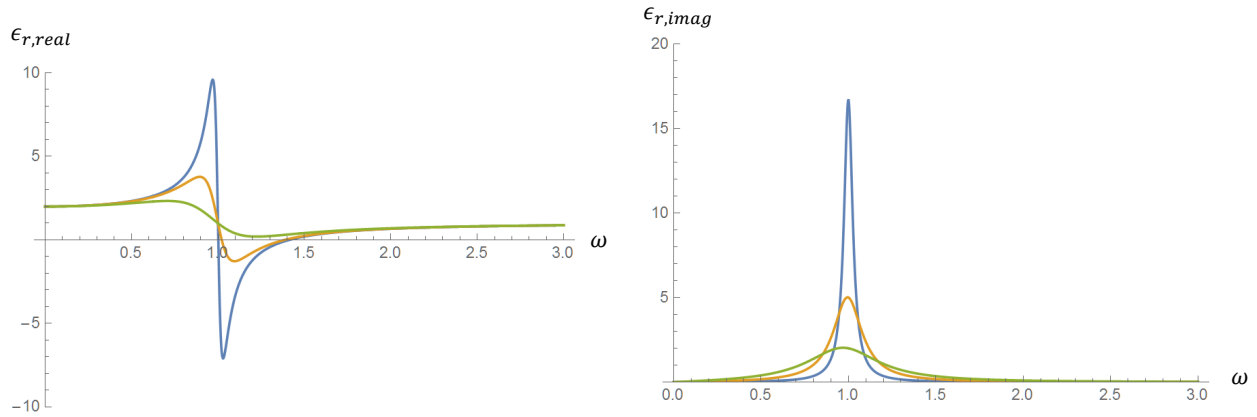
And because the susceptibility and the permittivity (aka dielectric constant) are related via  $\epsilon_r = 1 + \chi$ ,

$$\boxed{\tilde{\epsilon}_r = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}} \quad \text{Complex permittivity}$$

Note that even in this fairly straightforward model we arrive at a *complex* permittivity, which leads to both complex wave vectors and indices of refraction as per the “complex wave vector” handout. And more than that, it also predicts that the permittivity (and hence index of refraction) should have frequency dependence: there is *dispersion*.

### Plots of Lorentz oscillator model permittivity

Here's what the real and imaginary parts of that equation for  $\tilde{\epsilon}_r$  look like as a function of  $\omega$ , plotted with  $\omega_p = \omega_0 = 1$  for simplicity; each of the two plots includes three values of  $\gamma$ : 0.06, 0.2, and 0.5. As the damping decreases, the peaks get narrower and taller.



**Colton plots.** Plots of  $\epsilon_{r,real}$  and  $\epsilon_{r,imag}$  for a material with an electronic resonance, plotted with  $\omega_0 = 1$  and three different damping values.

Notice how regardless of what the damping constant might be,  $\tilde{\epsilon}_r$  goes to a specific low frequency (DC) value, and to a different specific high frequency value. Those values can be obtained by using  $\omega = 0$  and  $\omega = \infty$  in the equation.

Low frequency value:

$$\tilde{\epsilon}_r(\text{DC}) = 1 + \frac{\omega_p^2}{\omega_0^2 - 0^2 - i0\gamma}$$

$$\tilde{\epsilon}_r(\text{DC}) = 1 + \frac{\omega_p^2}{\omega_0^2}$$

High frequency value:

$$\tilde{\epsilon}_r(\text{high freq}) = 1 + \frac{\omega_p^2}{\omega_0^2 - \infty^2 - i\infty\gamma}$$

$$\tilde{\epsilon}_r(\text{high freq}) = 1$$

The low and high frequency values are real. It is only in the neighborhood of the resonance ( $\omega$  close to  $\omega_0$ ) that the complex nature of the permittivity is important.

Plots of the Lorentz model susceptibility would be essentially the same, just with the real part shifted down by one, since  $\chi = \epsilon_r - 1$ .

### Oscillator strength

To generalize this a bit more, we can allow for different types of electrons by introducing a summation:

$$\tilde{\epsilon}_r = 1 + \sum_i \frac{\omega_{p,i}^2}{\omega_{0,i}^2 - \omega^2 - i\omega\gamma_i}$$

Here  $\omega_{p,i}$ ,  $\omega_{0,i}$  and  $\gamma_i$  are the plasma frequency, resonant frequency, and damping coefficient, respectively, for the  $i^{\text{th}}$  type of electron.

By introducing the *oscillator strength*,  $f_i$ , which describes the fraction of electrons of type  $i$ , namely

$$f_i = N_i/N_{tot}$$

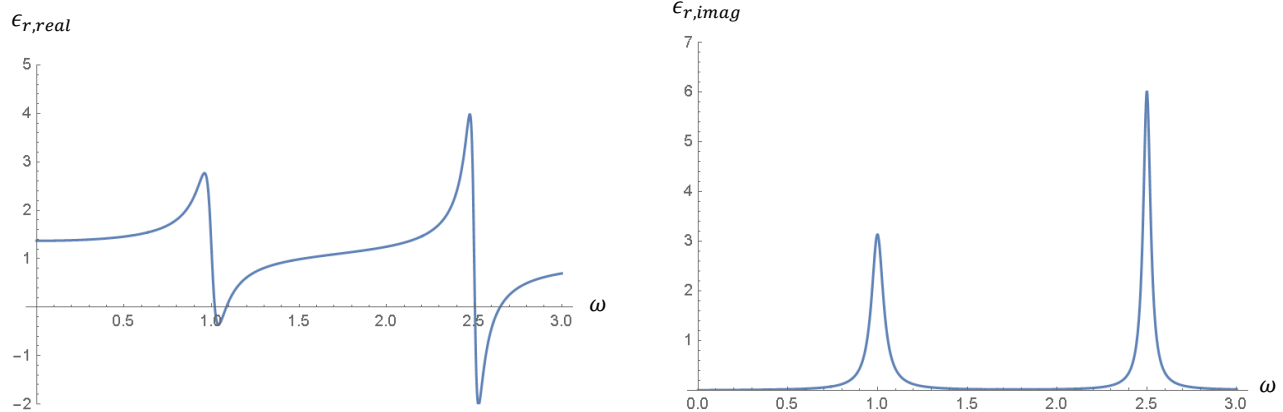
the equation can then be written in final form:

$$\epsilon_r = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Generalized Lorentz oscillator model permittivity

The plasma frequency here is defined by the *total* number of oscillating electrons per volume.

Here are some representative plots for the situation of two resonances, again giving the real and imaginary components of  $\epsilon_r$ , and again setting  $\omega_p = 1$  for simplicity. The first resonance has  $\omega_0 = 1$ ,  $\gamma = 0.08$ , and  $f = 0.25$ ; the second resonance has  $\omega_0 = 2.5$ ,  $\gamma = 0.05$ , and  $f = 0.75$ .



**Colton plots.** Plots of  $\epsilon_{r,real}$  and  $\epsilon_{r,imag}$  for an material with two electronic resonances having different  $\omega_0$ , oscillator strengths, and damping values..

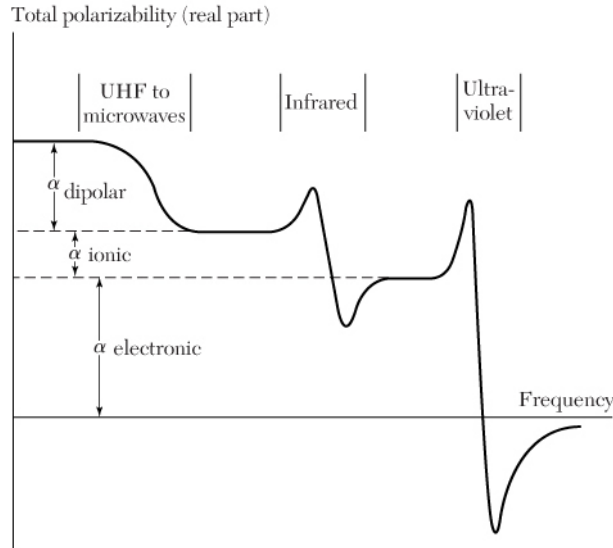
### Applied to a gas, from Griffiths

Griffiths applies this model and the boxed equation for permittivity to a gas of molecules, but defines the quantities slightly differently. Instead of  $N$  being the total number of oscillating electrons and  $f_i$  being the fraction of electrons of type  $i$ , he uses  $N$  as the total number of *molecules* and  $f_i$  being the *number of oscillating electrons of type  $i$  within each molecule*. It amounts to the same thing, though, since the number of molecules  $\times$  number of oscillating electrons of type  $i$  within each molecule is the same as the number of oscillating electrons  $\times$  fraction of electrons of type  $i$ .

Additionally, since gases are dilute,  $N$  will be small, and the terms in the summation will be much less than one. Therefore the index of refraction can be approximated by  $\tilde{n} = \sqrt{\epsilon_r} = \sqrt{1 + \text{small}} \approx 1 + \frac{1}{2}(\text{small})$ . Griffiths uses that approximation to obtain Eqs. 9.169-9.171 (both editions).

### Applied to insulating solids

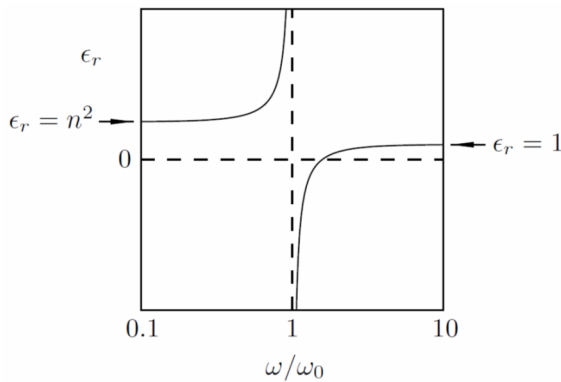
Here's a schematic plot from a solid state physics textbook by Kittel, describing how the real part of the polarizability (directly related to polarization) of atoms in solids changes with frequency. Aside from what happens at low frequencies, labeled as a dipolar effect, you should notice a lot of similarity with the two-resonance plot above. One resonance is labeled "electronic" and occurs in the ultraviolet; the other is labeled "ionic" and occurs in the infrared.



**Kittel, *Introduction to Solid State Physics*, Fig. 16-8.** Schematic of the frequency dependence of the several contributions to the polarizability.

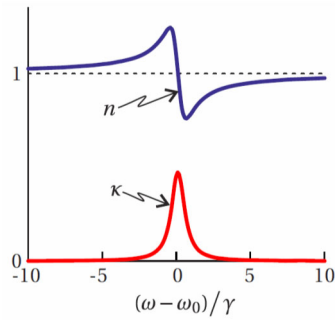
UV resonances: oscillations of electrons

The UV resonances most closely correspond to what we've been describing: electrons on "springs" oscillating back and forth in response to an applied field. Here's a plot of  $\epsilon_{r,real}$  from this model from a solid state physics book by Stokes (retired BYU professor), assuming a single resonance and no damping at all ( $\gamma = 0$ ).



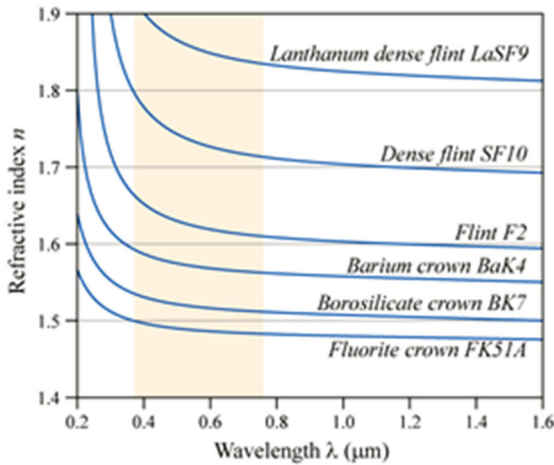
**Stokes, *Solid State Physics for Advanced Undergraduate Students*, Fig. 16-5.** The dielectric constant near the resonant frequency of the orbital electrons in a solid, using the Lorentz model but with no damping. He is using the letter  $n$  to refer to the low frequency value of the square root of  $\epsilon_r$  (which is not standard).

The Peatross and Ware Optics textbook also applies this model to solids and has a similar plot; in this case they plot the real and imaginary parts of  $\tilde{n} = \sqrt{\tilde{\epsilon}}$ , labeled  $n$  and  $\kappa$ , respectively.



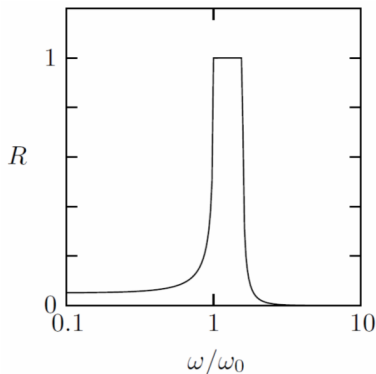
**Peatross and Ware, *Physics of Light and Optics*, Fig. 2.5.** Real and imaginary parts of the index of refraction for a single Lorentz oscillator dielectric with  $\omega_p = 10\gamma$ . [Note: this makes it look like  $n$  goes to 1 on the far left of the graph. That is not correct;  $n$  goes to a constant which is greater than 1.]

Keep in mind the resonance itself is typically at UV frequencies. The shape of  $n$  at frequencies below the resonance (i.e. visible wavelengths) explains the normal dispersion found in glasses. See for example this plot from Wikipedia.



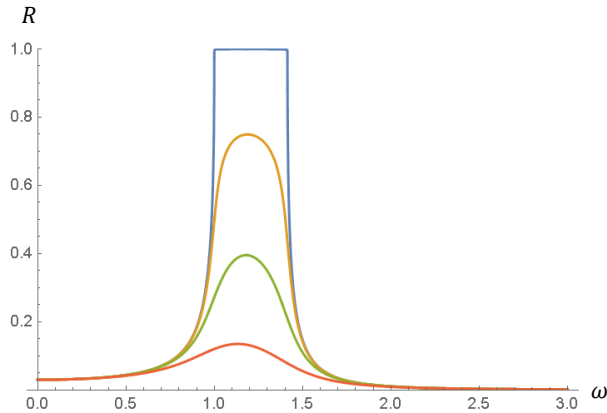
**Wikipedia, *Dispersion (optics)***  
This corresponds to the  $n$  vs  $\omega$  plot for frequencies just below the resonance (plotted here vs  $\lambda$  instead of  $\omega$ ).

The reflectivity can easily be calculated once the complex index of refraction is known, so Stokes has plotted here how the reflectivity in insulators changes with frequency near the resonance, again for the situation of a single resonance with no damping. Insulators become very reflective for frequencies just above  $\omega_0$ .



**Stokes, Fig. 16-6.** The reflectivity near the resonant frequency of the orbital electrons in a solid, at normal incidence and with no damping, plotted for an air/insulator interface using the equations  $\tilde{n} = \sqrt{\tilde{\epsilon}_r}$  and  $R = \left| \frac{1-\tilde{n}}{1+\tilde{n}} \right|^2$ .

Damping rounds out the corners and decreases the maximum reflectivity. Here's my own plot of  $R$  for several values of damping, namely  $\gamma = 0, 0.06, 0.2,$  and  $0.5$  (again with  $\omega_p = \omega_0 = 1$  for simplicity).

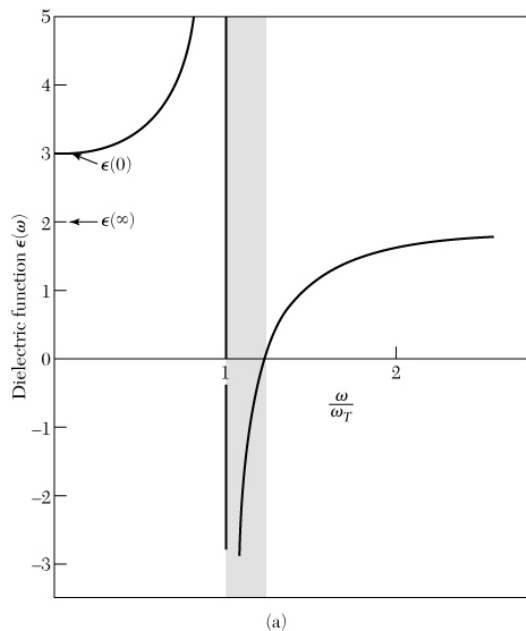


**Colton plot.** Plot of  $R$  for the UV resonance in an insulator, with  $\omega_p = 1$  and four different damping values.

### Infrared resonances: oscillations of ions

In ionic or partially ionic materials, the atoms themselves are charged, and their interactions with neighboring atoms cause them to be anchored to their lattice spots as with a spring. The Lorentz oscillator model can therefore be used to explain their behavior as well! The masses are much larger, so the resonant frequencies are much smaller. The relevant charge  $q$  is the ionic charge, which in general would be different than the charge of an electron,  $e$ . In partially ionic materials the charge can even be fractional.

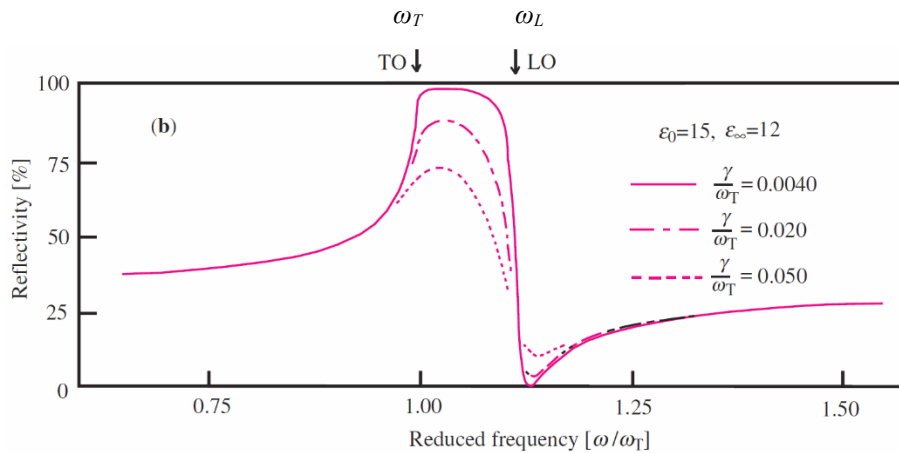
Here Kittel plots  $\epsilon_{r,real}$  from the Lorentz model for such oscillating ions.



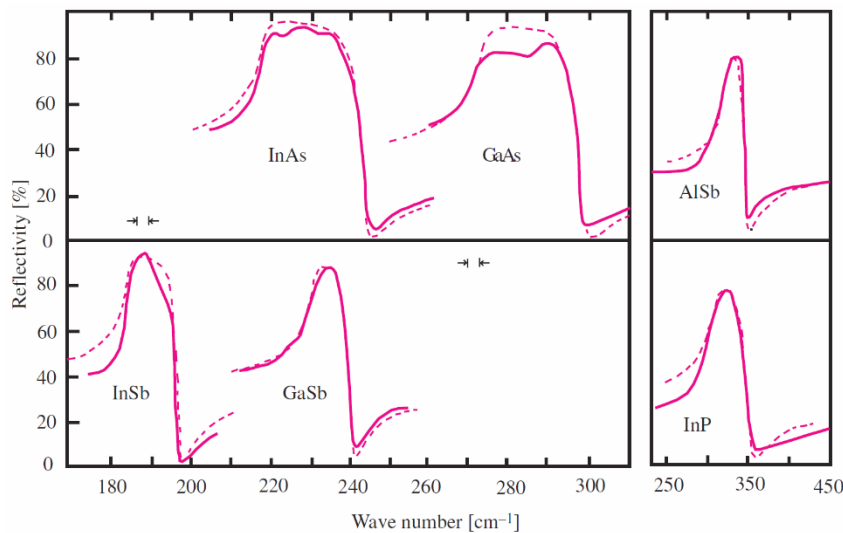
**Kittel, Fig. 14.13a.** The dielectric constant near the resonant frequency of the oscillating ions in a solid, using the Lorentz model with no damping. The right hand side doesn't go to 1 as the basic equation would permit; this is discussed a bit more below in the context of metals.  $\omega_T$  is defined to be the frequency where  $\omega$  crosses from positive infinity to negative infinity.  $\omega_L$  is defined to be the frequency where  $\epsilon_r$  goes positive again.  $T$  and  $L$  stand for transverse and longitudinal and relate to the types of phonon oscillations induced by the waves (beyond the scope of this course). Electromagnetic waves with frequencies in the shaded region, between  $\omega_T$  and  $\omega_L$ , will not propagate in the medium but instead will have 100% reflectivity (if no damping) just like is demonstrated in the plots for the UV resonance, above.

A semiconductor physics textbook by Yu and Cardona plots the results of this model applied to ions, along with some actual experimental data for real materials.





**Yu & Cardona, Fundamentals of Semiconductors, Fig. 6.31(b).** The infrared reflectivity from ionic oscillations, calculated from the Lorentz model with a few different values of damping.



**Yu & Cardona, Fig. 6.32.** Actual experimental reflectivity data (solid curves), and theoretical fits from the Lorentz model (dashed curves) using  $\omega_T$ ,  $\omega_L$ , and  $\gamma$  as fitting parameters.

*Important note on units:* As you can see, the x-axis is labeled “Wave number [cm<sup>-1</sup>]”. CAUTION: that is not what we’ve been calling the wave number! The wave number  $k$ , as we’ve been using it, would be labeled as rad/m or rad/cm. By contrast, when you see experimental data that is labeled “cm<sup>-1</sup>”, particularly with optical data like this, they nearly always mean  $1/\lambda$  instead of  $2\pi/\lambda$ . I believe in the “olden days”  $k$  was originally defined as  $1/\lambda$ , and this has persisted in some settings (like this) even today. If you see a feature on a graph like this at, say, 185 cm<sup>-1</sup>, you can convert it to regular wavelength like this:

$$185 \text{ cm}^{-1} = 18500 \text{ m}^{-1} \rightarrow \text{take inverse, } \lambda = 5.405 \times 10^{-5} \text{ meters} \approx 54 \text{ } \mu\text{m}$$

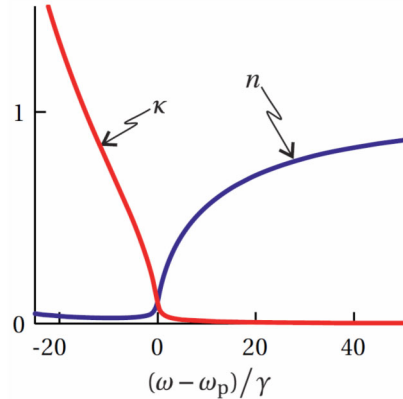
### Applied to metals

The fun doesn’t stop! The valence electrons in metals are not anchored to their nuclei (no springs) but instead are free to move around the material. However, they still respond to the electric field and experience damping in the same way as the electrons in insulators for which we derived the Lorentz oscillator model above. Can we just set  $\omega_0 = 0$  to account for no restoring forces? You bet we can! Setting  $\omega_0 = 0$  in the Lorentz oscillator model (which, with no restoring forces is sometimes called the Drude model) trivially results in:

$$\tilde{\epsilon}_r = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma}$$

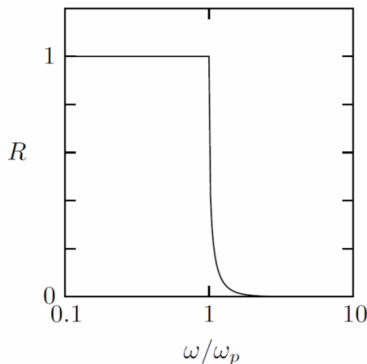
Complex permittivity, metals

Peatross and Ware plot the real and imaginary parts of  $\tilde{n} = \sqrt{\tilde{\epsilon}_r}$  for this situation.

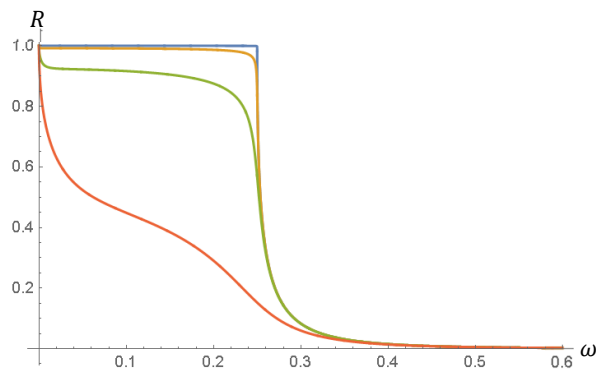


**Peatross and Ware, Fig. 2.6.** Real and imaginary parts of the index for a conductor with  $\gamma = 0.02 \omega_p$ .

Stokes uses this equation for  $\tilde{\epsilon}_r$  to obtain  $\tilde{n}$  and then  $R$  for normal incidence, which is plotted. You can also see my own plot which includes a few additional values of  $\gamma$ , namely  $\gamma = 0, 0.001, 0.01, 0.1$  (I've set  $\omega_p = 0.25$  for simplicity).



**Stokes, Fig. 16-10.** The reflectivity of metal near its plasma frequency, with no damping.



**Colton plot.** Plot of  $R$  for a metal with  $\omega_p = 0.25$  and four different damping values.

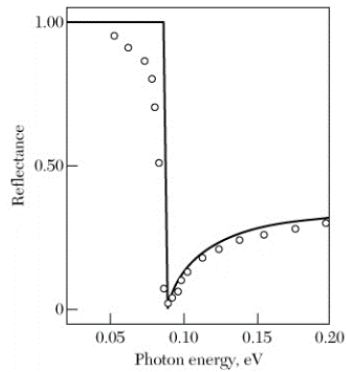
Metals are highly reflective for frequencies below their specific plasma frequencies. In most metals the plasma frequency is in the UV; therefore they reflect visible light very efficiently. On the other hand, metals become very transparent for frequencies above the plasma frequency.

Finally, as mentioned above in the caption to Kittel, Fig. 14.13a, in some materials the high frequency dielectric constant does not go to 1. To account for  $\epsilon_r$  not going to one, we can simply change the 1 in our complex permittivity equation for metals to a different constant called  $\epsilon_\infty$ , such that

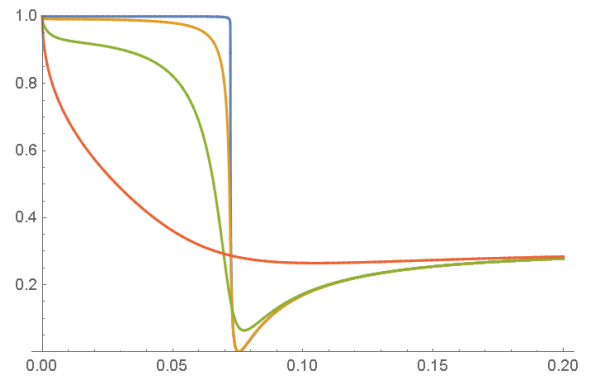
$$\tilde{\epsilon}_r = \epsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}$$

Here's a plot from Kittel using that equation to calculate and fit the reflectivity of indium antimonide (InSb), also with my own plot of  $R$  on the right, calculated by using that modified equation for  $\tilde{\epsilon}_r$  to obtain  $\tilde{n}$ . In my plot I've used  $\epsilon_\infty = 12$ ,  $\omega_p = 0.25$ , and four damping values:  $\gamma = 0, 0.001, 0.01, 0.1$ . Notice how different

my plot here is compared to the one just above, in terms of where the reflectivity abruptly dips, even though  $\omega_p$  is still 0.25.



**Kittel, Fig. 14.3.** Experimental reflectivity of InSb (empty points), fitted with the Lorentz model with no damping (solid line). InSb is a narrow band gap semiconductor which acts fairly metallic at room temperature.



**Colton plot.** Plot of  $R$  with  $\epsilon_\infty = 12$ ,  $\omega_p = 0.25$ , and four different damping values. This plot makes it look like Kittel should have used some damping in the fit on the left to smooth out the sharp features in his solid line.