ECE 5322
21st Century Electromagnetics

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Lecture #2

Electromagnetic Properties of Materials – Part I

Lorentz and Drude Models

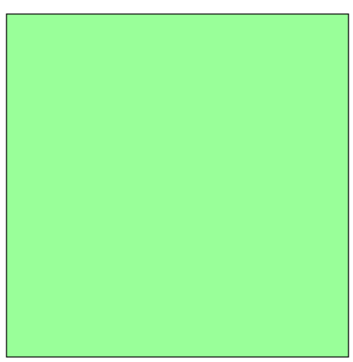
Lecture Outline

- High level picture of dielectric response
- Resonance
- Lorentz model for dielectrics
- Lorentz model for permeability
- Drude model for metals
- Generalizations
- Other materials models

High Level Picture of Dielectric Response

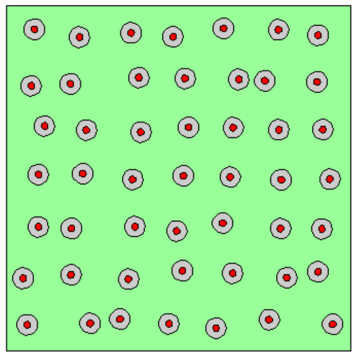
Dielectric Slab

We wish to understand why a dielectric exhibits an electromagnetic response.



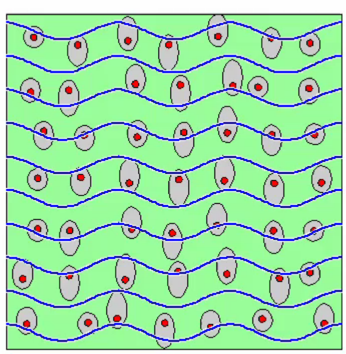
Atoms at Rest

Without an applied electric field, the electron “clouds” around the nuclei are symmetric and at rest.



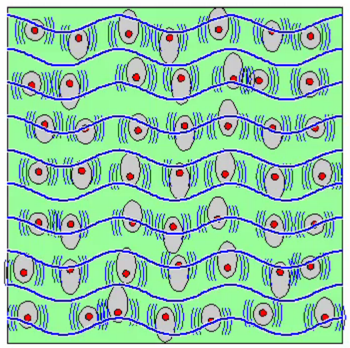
Applied Wave

The electric field of a electromagnetic wave pushes the electrons away from the nuclei producing “clouds” that are offset.



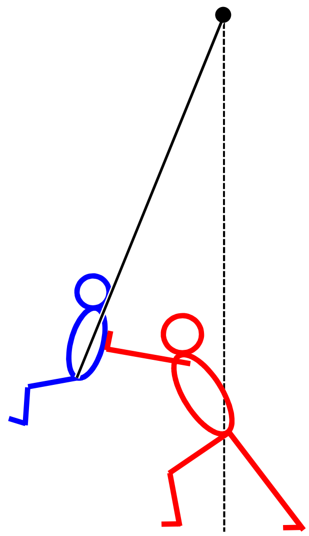
Secondary Waves

The motion of the charges emits secondary waves that interfere with the applied wave to produce an overall slowing effect on the wave.



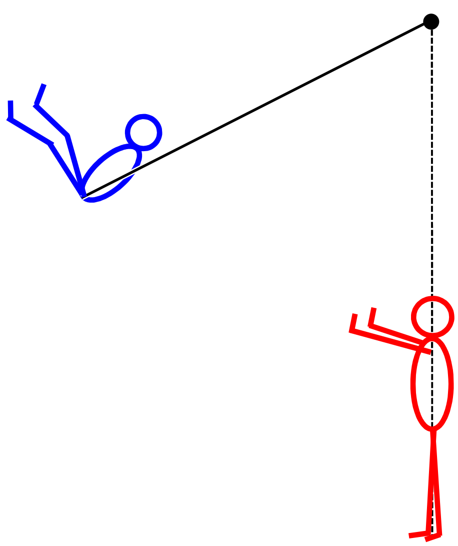
Resonance

Visualizing Resonance – Low Frequency



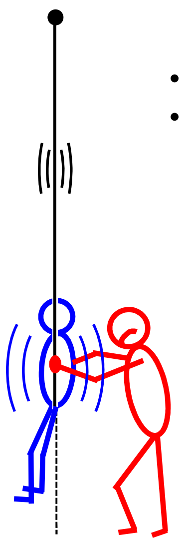
- Can push object to modulate amplitude
- Displacement is in phase with driving force
- DC offset

Visualizing Resonance – on Resonance



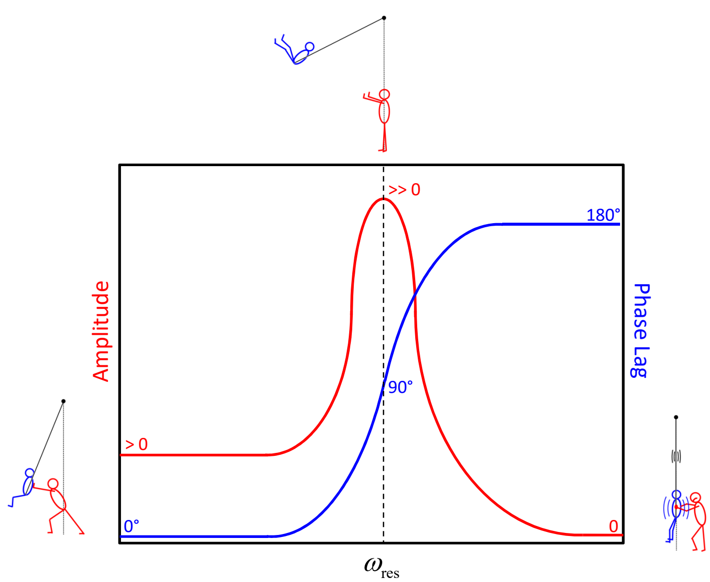
- Can push object to large amplitude
- Displacement and driving force are 90° out of phase
  - Peaks of push correspond to nulls of displacement

Visualizing Resonance – High Frequency

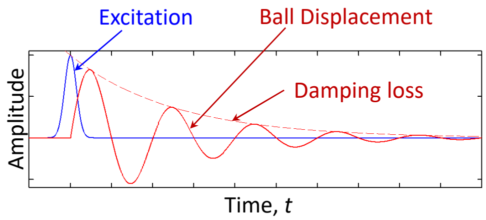


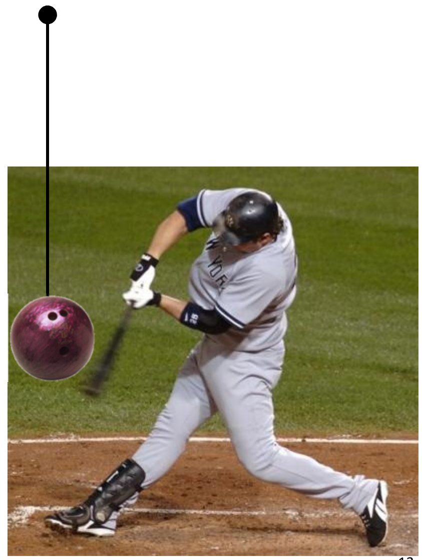
- Vanishing amplitude
- Displacement is 180° out of phase

A Harmonic Oscillator

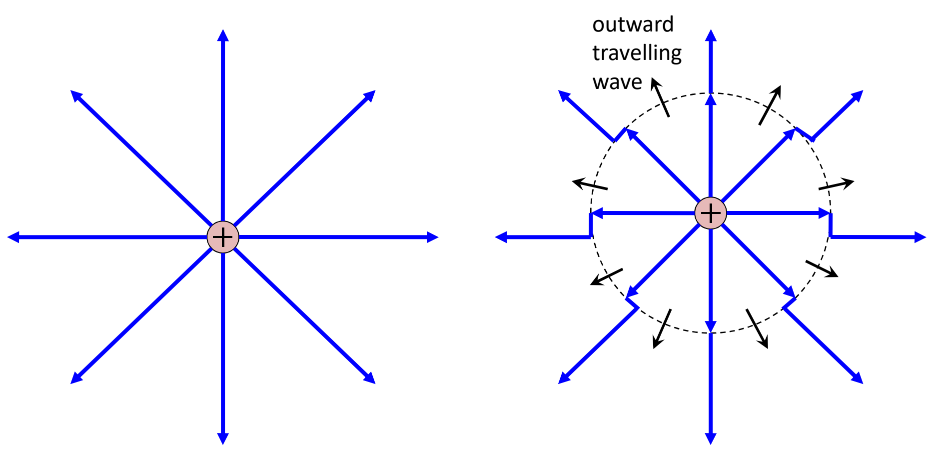


Impulse Response of a Harmonic Oscillator



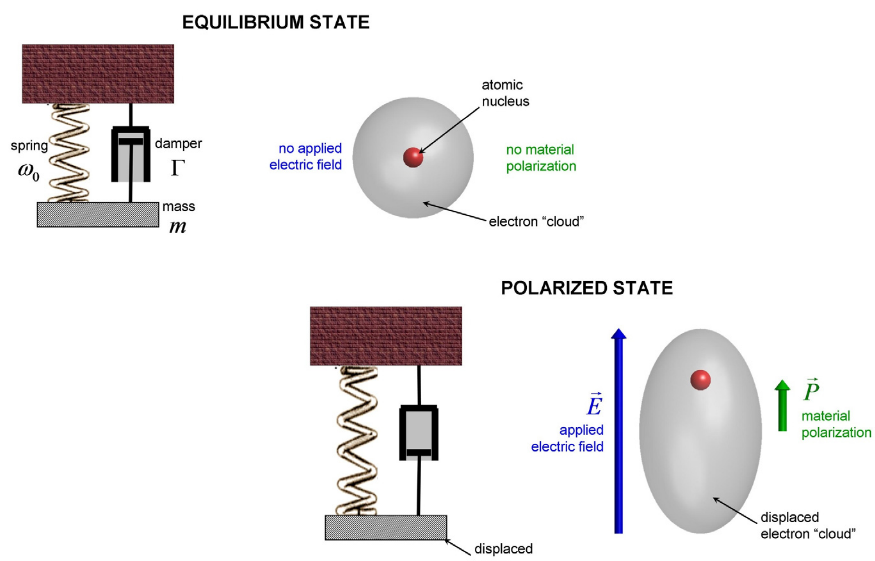


Moving Charges Radiate Waves



Lorentz Model for Dielectrics

Lorentz Oscillator Model



Maxwell's Equations with Material Polarization

Material polarization is incorporated into the constitutive relations.

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Response of material

Response of free space

Constitutive relation in terms of relative permittivity and susceptibility.

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon}_r \vec{E} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E}$$

Comparing the above equations, we see that

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} \quad \rightarrow \quad \vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon}_r \vec{E} = \varepsilon_0 \left(1 + \chi\right) \vec{E} \quad \rightarrow \quad \tilde{\varepsilon}_r = 1 + \chi$$

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Equation of Motion

$$m \frac{\partial^2 \vec{r}}{\partial t^2} + m\Gamma \frac{\partial \vec{r}}{\partial t} + m\omega_0^2 \vec{r} = -q\vec{E}$$

acceleration force

$m \Rightarrow m_e$ mass of an electron

frictional force

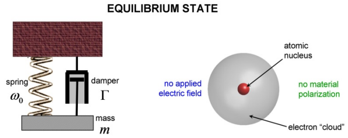
Γ damping rate (loss/sec)

restoring force

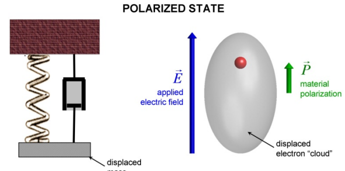
$\omega_0 = \sqrt{\frac{K}{m}}$ natural frequency

electric force

EQUILIBRIUM STATE



POLARIZED STATE



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Fourier Transform

$$m\frac{\partial^2 \vec{r}}{\partial t^2} + m\Gamma\frac{\partial \vec{r}}{\partial t} + m\omega_0^2\vec{r} = -q\vec{E}$$

$$\Big\downarrow \text{ Fourier transform}$$

$$m(-j\omega)^2\,\vec{r}(\omega) + m\Gamma(-j\omega)\vec{r}(\omega) + m\omega_0^2\vec{r}(\omega) = -q\vec{E}(\omega)$$

$$\Big\downarrow \text{ Simplify}$$

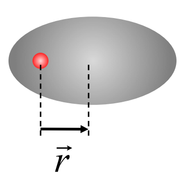
$$\left(-m\omega^2 - j\omega m\Gamma + m\omega_0^2\right)\vec{r}(\omega) = -q\vec{E}(\omega)$$

Displacement

$$\left(-m\omega^2 - j\omega m\Gamma + m\omega_0^2\right)\vec{r}(\omega) = -q\vec{E}(\omega)$$

$$\Big\downarrow$$

$$\vec{r}(\omega) = -\frac{q}{m_e}\frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$



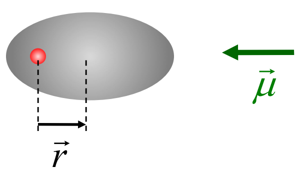
Dipole Moment

Definition of Dipole Moment: $\vec{\mu}(\omega) = -q\vec{r}(\omega)$

charge distance from center

** Sorry for the confusing notation, but μ here is NOT permeability.

$$\vec{\mu}(\omega) = \frac{q^2}{m_e} \frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$



Lorentz Polarizability, α

Definition of Polarizability: $\vec{\mu}(\omega) = [\alpha(\omega)]\vec{E}(\omega)$

** Sorry for the confusing notation, but α here is NOT absorption.

$\alpha(\omega)$ is a tensor quantity for anisotropic materials.
For simplicity, we will use the scalar form.
This is the Lorentz polarizability for a single atom.

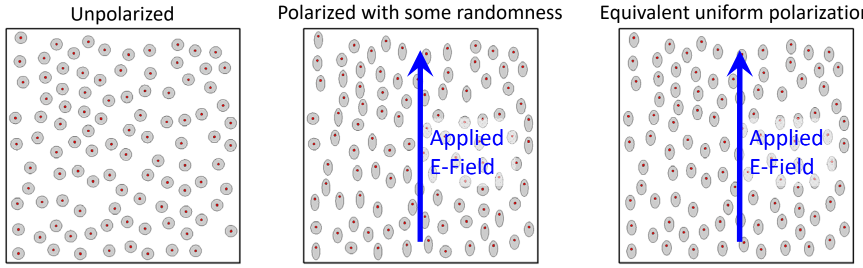
$$\alpha(\omega) = \frac{q^2}{m_e} \frac{1}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

Polarization per Unit Volume

Definition: $\vec{P}(\omega)=\frac{1}{V}\sum_{V}\vec{\mu}_i(\omega)$

Average dipole moment over all atoms in a material.
All billions and trillions of them!!!

Unpolarized | Polarized with some randomness | Equivalent uniform polarization



There is some randomness to the polarized atoms so a statistical approach is taken to compute the average.

$$\vec{P}(\omega)=N\left\langle\vec{\mu}(\omega)\right\rangle$$

$N \equiv$ Number of atoms per unit volume
$\langle\ \rangle \equiv$ Statistical average

Susceptibility (1 of 2)

Recall the following:

$$\vec{P}(\omega)=N\left\langle\vec{\mu}(\omega)\right\rangle=\varepsilon_0\chi(\omega)\vec{E}(\omega)$$

$$\vec{\mu}(\omega)=\alpha(\omega)\vec{E}(\omega)$$

$$\alpha(\omega)=\frac{q^2}{m_e}\frac{1}{\omega_0^2-\omega^2-j\omega\Gamma}$$

This leads to an expression for the susceptibility:

$$\chi(\omega)=\frac{N\alpha(\omega)}{\varepsilon_0}=\left(\frac{Nq^2}{\varepsilon_0 m_e}\right)\frac{1}{\omega_0^2-\omega^2-j\omega\Gamma}$$

Susceptibility (2 of 2)

Susceptibility of a dielectric material:

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

$$\omega_p^2 = \frac{Nq^2}{\varepsilon_0 m_e}$$ plasma frequency

$$q = 1.60217646 \times 10^{-19} \text{ C}$$
$$\varepsilon_0 = 8.8541878176 \times 10^{-12} \text{ F/m}$$
$$m_e = 9.10938188 \times 10^{-31} \text{ kg}$$

- Note this is the susceptibility of a dielectric which has only one resonance.
- The location of atoms is important because they can influence each other. We ignored this.
- Real materials have many sources of resonance and all of these must be added together.

The Dielectric Function

Recall that,

$$\vec{D} = \varepsilon_0 \tilde{\varepsilon}_r \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (1 + \chi) \vec{E}$$

Therefore,

$$\tilde{\varepsilon}_r(\omega) = \underbrace{1}_{\text{vacuum}} + \underbrace{\chi(\omega)}_{\text{material}}$$

The ~ symbol indicates the quantity is complex

The dielectric function for a material with a single resonance is then,

$$\boxed{\tilde{\varepsilon}_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma} \qquad \omega_p^2 = \frac{Nq^2}{\varepsilon_0 m_e}}$$

Summary of Derivation

1. We wrote the equation of motion by comparing bound charges to a mass on a spring.
$$m_e \frac{\partial^2 \vec{r}}{\partial t^2} + m_e \Gamma \frac{\partial \vec{r}}{\partial t} + m_e \omega_0^2 \vec{r} = -q\vec{E}$$
2. We performed a Fourier transform to solve this equation for ***r***.
$$\vec{r}(\omega) = -\frac{q}{m_e} \frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$
3. We calculated the electric dipole moment of the charge displaced by ***r***.
$$\vec{\mu}(\omega) = \frac{q^2}{m_e} \frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$
4. We calculated the volume averaged dipole moment to derive the material polarization.
$$\vec{P}(\omega) = \tfrac{1}{V} \sum \vec{\mu}_i(\omega) = N \langle \vec{\mu}(\omega) \rangle$$
5. We calculated the material susceptibility.
$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma} \qquad \omega_p^2 = \frac{Nq^2}{\varepsilon_0 m_e}$$
6. We calculated the dielectric function.
$$\boxed{\tilde{\varepsilon}_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}}$$

Real and Imaginary Parts of ε

$$\boxed{\tilde{\varepsilon}_r(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma} \qquad \omega_p^2 = \frac{Nq^2}{\varepsilon_0 m_e}}$$

Split into real and imaginary parts

$$\begin{aligned}
\tilde{\varepsilon}_r(\omega) = \varepsilon_r'(\omega) + j\varepsilon_r''(\omega) &= 1 + \frac{\omega_p^2}{\left(\omega_0^2 - \omega^2\right) - j\omega\Gamma} \frac{\left(\omega_0^2 - \omega^2\right) + j\omega\Gamma}{\left(\omega_0^2 - \omega^2\right) + j\omega\Gamma} \\
&= 1 + \omega_p^2 \frac{\left(\omega_0^2 - \omega^2\right) + j\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \\
&= 1 + \omega_p^2 \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} + j\omega_p^2 \frac{\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}
\end{aligned}$$

$$\boxed{\varepsilon_r'(\omega) = 1 + \omega_p^2 \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \qquad \varepsilon_r''(\omega) = \omega_p^2 \frac{\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}}$$

Complex Refractive Index $\tilde{n}$

Refractive is like a "density" to an electromagnetic wave. It quantifies the speed of an electromagnetic wave through a material. Waves travel slower through materials with higher refractive index.

$$\tilde{n} = n + j\kappa = \pm\sqrt{\tilde{\mu}_r \tilde{\varepsilon}_r} = \pm\sqrt{(1+\chi_m)(1+\chi_e)}$$

For now, we will ignore the magnetic response.

$$\tilde{n} = n + j\kappa = \pm\sqrt{\tilde{\varepsilon}_r}$$

$n \equiv$ ordinary refractive index

$\kappa \equiv$ extinction coefficient

Converting between dielectric function and refractive index.

$$n + j\kappa = \pm\sqrt{\varepsilon_r' + j\varepsilon_r''}$$

$$(n + j\kappa)^2 = \varepsilon_r' + j\varepsilon_r''$$

$$n^2 + jn\kappa + jn\kappa - \kappa^2 = \varepsilon_r' + j\varepsilon_r''$$

$$(n^2 - \kappa^2) + j2n\kappa = \varepsilon_r' + j\varepsilon_r''$$

$\tilde{n} \rightarrow \varepsilon$

$$\varepsilon_r' = n^2 - \kappa^2$$

$$\varepsilon_r'' = 2n\kappa$$

$\varepsilon \rightarrow \tilde{n}$

$$n + j\kappa = \pm\sqrt{\varepsilon_r' + j\varepsilon_r''}$$

Absorption Coefficient (1 of 2)

From Maxwell's equations, a plane wave in a linear, homogeneous, isotropic (LHI) medium as…

$$\vec{E}(z) = \vec{E}_0 e^{jkz}$$

The wave number is

$$k = k_0 \tilde{n} \qquad k_0 = \frac{2\pi}{\lambda_0}$$

Substituting the complex refractive index into this equation leads to…

$$\vec{E}(z) = \vec{E}_0 e^{jk_0(n+j\kappa)z} = \vec{E}_0 \underbrace{e^{-k_0\kappa z}}_{\text{envelope term}} \underbrace{e^{jk_0 nz}}_{\text{Oscillatory term}}$$

The absorption coefficient α is defined in terms of the field intensity.

$$I(z) = I_0 e^{-\alpha z}$$

Absorption Coefficient (2 of 2)

The field intensity is related to the field amplitude through

$$I(z) = |E(z)|^2$$

Substituting expressions from the previous slide, the absorption coefficient can be calculated from κ

$$I(z) = |E(z)|^2$$

$$I_0 e^{-\alpha z} = \left| \vec{E}_0 e^{-k_0 \kappa z} e^{j k_0 n z} \right|^2$$

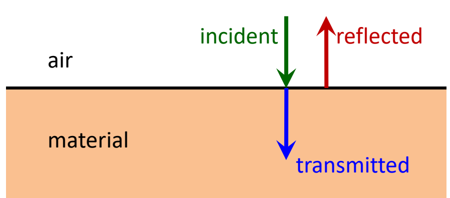
$$I_0 e^{-\alpha z} = \left| \vec{E}_0 \right|^2 e^{-2 k_0 \kappa z}$$

$$e^{-\alpha z} = e^{-2 k_0 \kappa z}$$

$$\alpha = 2 k_0 \kappa$$

$$\boxed{\alpha = 2 k_0 \kappa = \frac{2\omega}{c_0} \kappa}$$

Reflectance (normal incidence)



The amplitude reflection coefficient r quantifies the amplitude and phase of reflected waves.

$$r(\omega) = \frac{1 - n(\omega) - j\kappa(\omega)}{1 + n(\omega) + j\kappa(\omega)}$$

Loss contributes to reflections!

The power reflection coefficient (reflectance) is always positive and between 0 and 1 (for materials without gain).

$$R(\omega) = r(\omega) r^*(\omega) = \frac{\left[1 - n(\omega)\right]^2 + \kappa^2(\omega)}{\left[1 + n(\omega)\right]^2 + \kappa^2(\omega)}$$

Kramers-Kroenig Relations (1 of 3)

The susceptibility is essentially the impulse response of a material to an applied electric field.

$$\vec{E}(t) \longrightarrow \boxed{\varepsilon_0 \chi(t)} \longrightarrow \vec{P}(t)$$

$$\vec{P}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \vec{E}(\tau)\,\chi_e(t-\tau)\,d\tau$$

$$\vec{P}(\omega) = \varepsilon_0 \chi(\omega)\,\vec{E}(\omega)$$

Causality requires that $\chi(t=0)=P(t=0)=0$

From linear system theory, if $\chi(t)$ is a causal, than the real and imaginary parts of its Fourier transform are Hilbert transform pairs.

$$\chi(\omega) = \chi'(\omega) + j\chi''(\omega) \longrightarrow$$

$$\chi'(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi''(\Omega)}{\omega-\Omega}\,d\Omega$$

$$\chi''(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi'(\Omega)}{\omega-\Omega}\,d\Omega$$

This means that χ' and χ'' are not independent.
If we can measure one, we can calculate the other.

But how do we measure at negative frequencies?

Kramers-Kroenig Relations (2 of 3)

From Fourier theory, if $\chi(t)$ is purely real then

$\chi'(\omega)$ is an even function
$\chi''(\omega)$ is an odd function

Applying this symmetry principle to the relations on the previous page leads to

$$\chi'(\omega) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\Omega\chi''(\Omega)}{\Omega^2-\omega^2}\,d\Omega$$

$$\chi''(\omega) = -\frac{2\omega}{\pi}\int_{0}^{\infty} \frac{\chi'(\Omega)}{\Omega^2-\omega^2}\,d\Omega$$

These equations can be applied to measurements taken over just positive frequencies.

Kramers-Kroenig Relations (3 of 3)

For dilute media with weak susceptibility (χ' and χ'' are small), the complex refractive index can be approximated from the susceptibility as…

$$\tilde{n} = \sqrt{1+\chi} = \sqrt{1+\chi' + j\chi''} \approx 1 + \frac{\chi'}{2} + j\frac{\chi''}{2}$$

Comparing the real and imaginary components of $\tilde{n}$ and χ leads to the Kramers-Kroenig relations for the refractive index and absorption coefficient.

$$\boxed{n(\omega) \cong 1 + \frac{c_0}{\pi}\int_0^\infty \frac{\alpha(\Omega)}{\Omega^2 - \omega^2}\,d\Omega}$$

$$\boxed{\alpha(\omega) \cong \frac{c_0}{\pi}\int_0^\infty \frac{n(\Omega)-1}{\Omega^2 - \omega^2}\,d\Omega}$$

For dilute media

Summary of Properties

Dielectric Function

- The dielectric function is most fundamental to Maxwell's equations.
- Imaginary part only exists if there is loss. When there is loss, the real part contributes.
- Perhaps more difficult to extract physical meaning from the real and imaginary parts.

$$\varepsilon_r'(\omega) = 1 + \omega_p^2 \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} \qquad \varepsilon_r''(\omega) = \omega_p^2 \frac{\omega\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2}$$

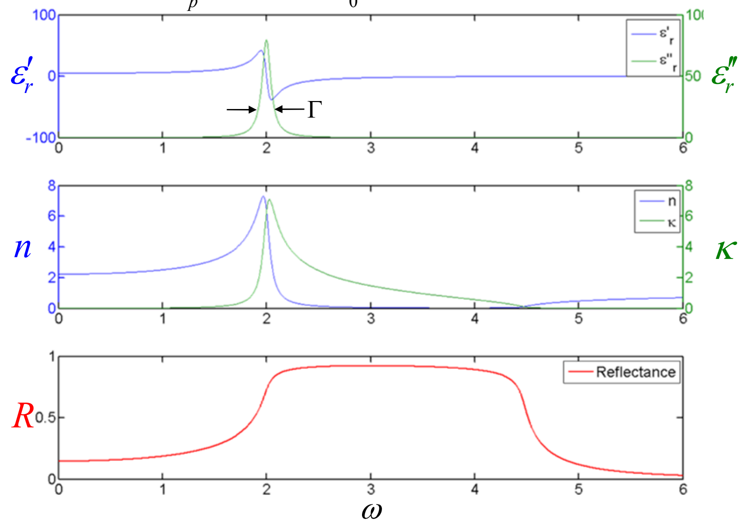
Refractive Index

- The refractive index is more closely related to wave propagation. It quantifies both velocity and loss.
- Real part is solely related to phase velocity.
- Imaginary part is solely related to loss.
- In many ways, refractive index is a more physically meaningful parameter.

$$\tilde{n} = n + j\kappa = \pm\sqrt{\tilde{\mu}_r \tilde{\varepsilon}_r} = \pm\sqrt{\left(1+\chi_m\right)\left(1+\chi_e\right)}$$

Typical Lorentz Model for Dielectrics

$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$



Example – Salt Water

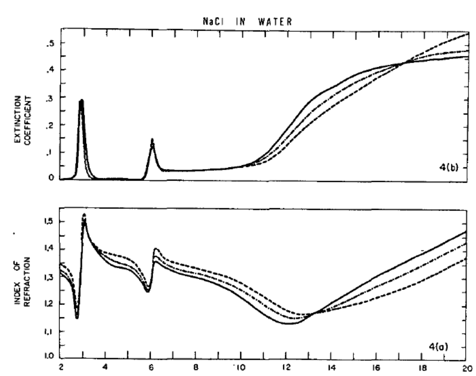


FIG. 4. The optical constants in the infrared for $1M$——, $3M$—·—, and $5M$— — — aqueous solutions of NaCl. The KK analysis and algorithm for computing these quantities are presented in the text along with estimates of the uncertainties of the optical constants.

M. R. Querry, R. C. Waring, W. E. Holland, M. Hale, W. Nijm, "Optical Constants in the Infrared for Aqueous Solutions of NaCl," J. Opt. Soc. Am. **62**(7), 849–855 (1972)

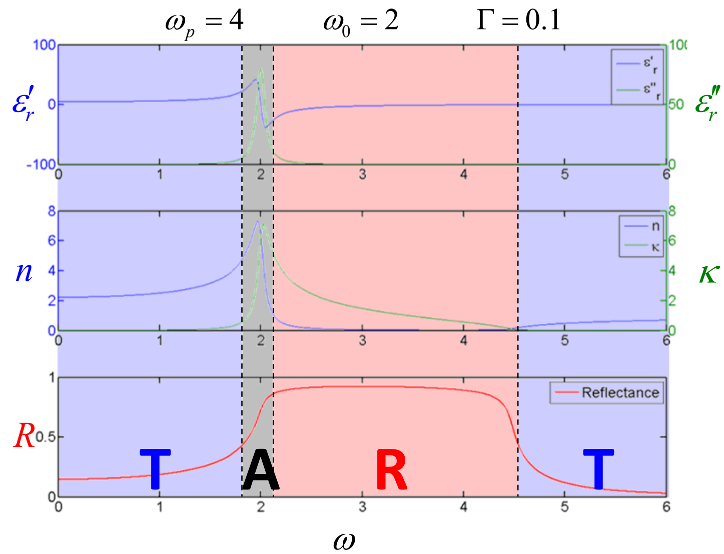
TART

Transmissive
$\omega < \omega_0 - \frac{\Gamma}{2}$

Absorptive
$\omega_0 - \frac{\Gamma}{2} < \omega < \omega_0 + \frac{\Gamma}{2}$

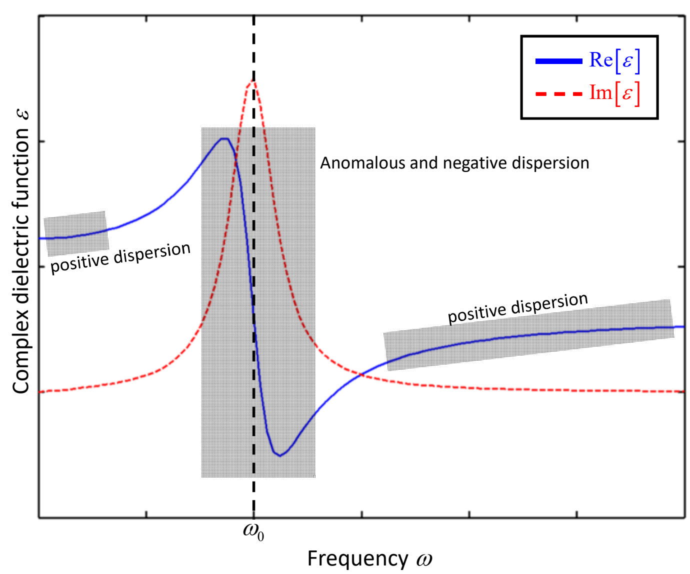
Reflective
$\omega_0 + \frac{\Gamma}{2} < \omega < \omega_p$

Transmissive
$\omega > \omega_p$



The TART regions are most distinct for small Γ and large ω_p

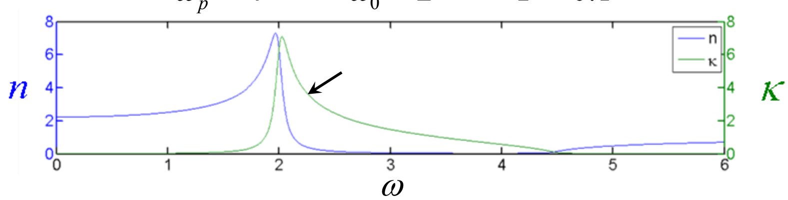
Dispersion



Observation #1

Loss is very high near resonance.

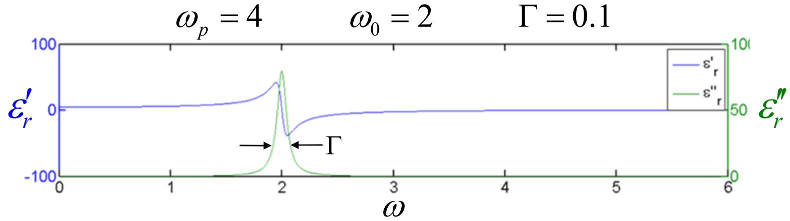
$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$



Observation #2

Damping rate determines width of resonance.

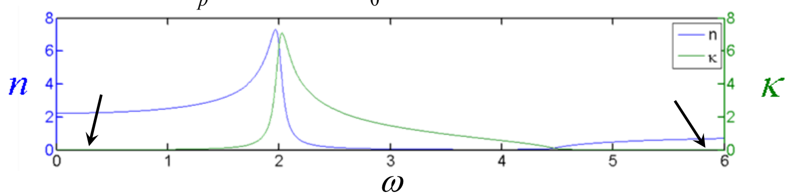
$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$



Observation #3

Far from resonance, loss is very low.

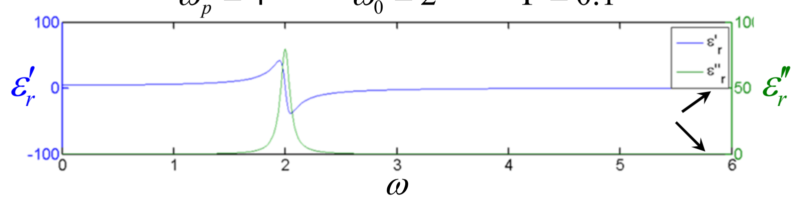
$$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$$



Observation #4

Material has no response at frequencies far above resonance.

$$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$$



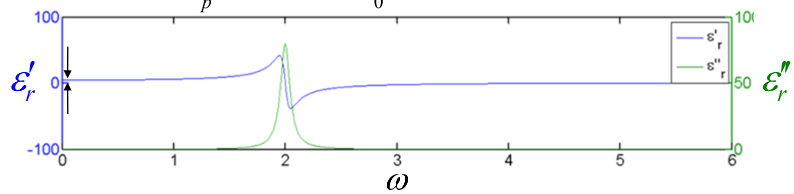
Materials, including metals, tend to become transparent at very high frequencies (e.g. x-rays).

Observation #5

Dielectric constant has a DC offset below resonance.

$$\varepsilon_r'(0) = 1 + \frac{\omega_p^2}{\omega_0^2}$$

$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$

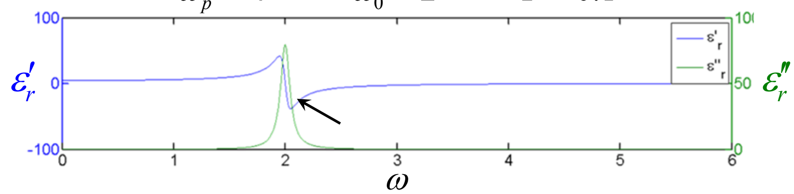


At frequencies well below the resonance, we can replace the Lorentz equation with just a simple constant.

Observation #6

Dielectric constant can be negative and/or less than one.

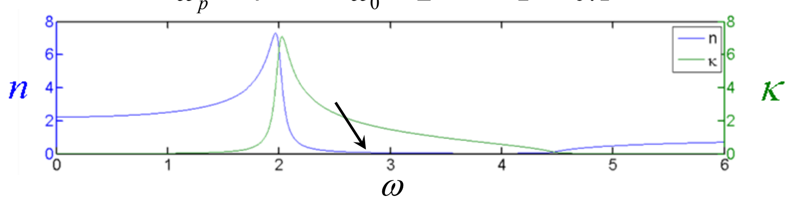
$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$



Observation #7

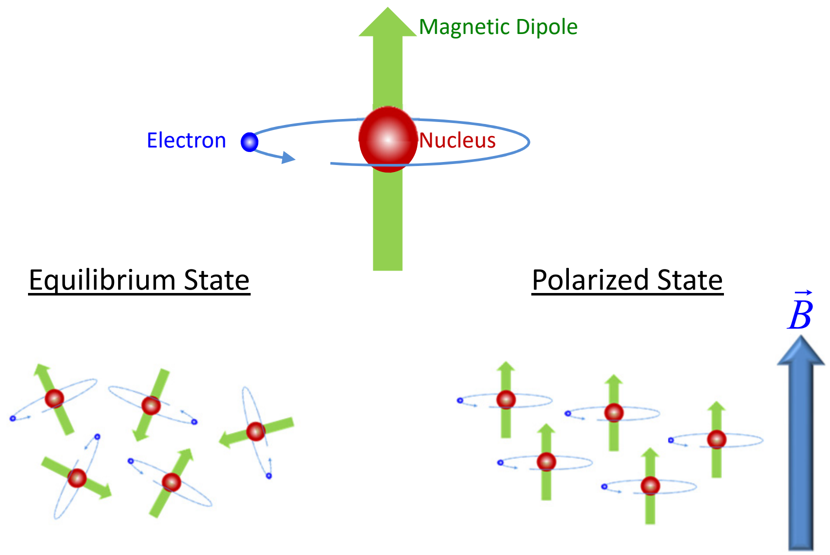
Refractive index can be less than one.

$\omega_p = 4 \qquad \omega_0 = 2 \qquad \Gamma = 0.1$



Lorentz Model for Permeability

Magnetic Response of Ordinary Materials



Lorentz Model for Permeability

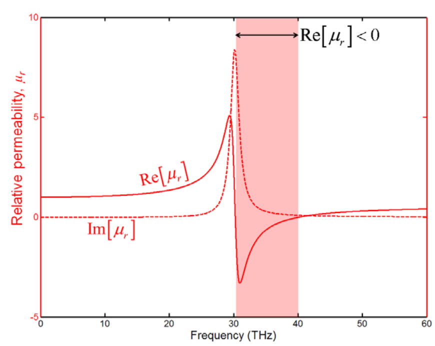
$$\tilde{\mu}_r(\omega) = 1 + \frac{\omega_{\mathrm{mp}}^2}{\omega_{\mathrm{m0}}^2 - \omega^2 - j\omega\Gamma_{\mathrm{m}}}$$

$\omega_{\mathrm{mp}} \equiv$ magnetic plasma frequency

$\omega_{\mathrm{m0}} \equiv$ magnetic resonant frequency

$\Gamma_{\mathrm{m}} \equiv$ magnetic damping rate

Boardman, Allan D., and Kiril Marinov, "Electromagnetic energy in a dispersive metamaterial," Phys. Rev. B, Vol. 73, No.16, pp. 165110, 2006.



Drude Model for Metals

Drude Model for Metals

In metals, most electrons are free because they are not bound to a nucleus. For this reason, the restoring force is negligible and there is no natural frequency.

We derive the Drude model for metals by assuming ω_0=0.

$$\tilde{\varepsilon}_r(\omega) = 1 + \frac{\omega_p^2}{\cancel{\omega_0^2} - \omega^2 - j\omega\Gamma} \qquad \omega_p^2 = \frac{Nq^2}{\varepsilon_0 m_e}$$

$$\boxed{\tilde{\varepsilon}_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\Gamma}}$$

Note, N is now interpreted as electron density N_e.

m_e is the effective mass of the electron.

Conductivity (1 of 2)

When describing metals, it is often more meaningful to put the equation in terms of the "mean collision rate" τ. This is also called the momentum scattering time.

$$\tilde{\varepsilon}_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + j\omega\tau^{-1}} \qquad \tau = \frac{1}{\Gamma}$$

This can be written in terms of the real and imaginary components.

$$\tilde{\varepsilon}_r = \left(1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2}\right) + j\left(\frac{\omega_p^2\tau/\omega}{1+\omega^2\tau^2}\right)$$

Conductivity (2 of 2)

In practice, metals are usually described in terms of a real-valued permittivity and a conductivity. These can be defined from above using Ampere's circuit law.

Ampere's Law with $\tilde{\varepsilon}_r$	Ampere's Law with ε_r and σ
$\nabla \times \vec{H} = j\omega\varepsilon_0\tilde{\varepsilon}_r\vec{E}$	$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon_0\varepsilon_r\vec{E}$

Comparing the two sets of Maxwell's equations leads to

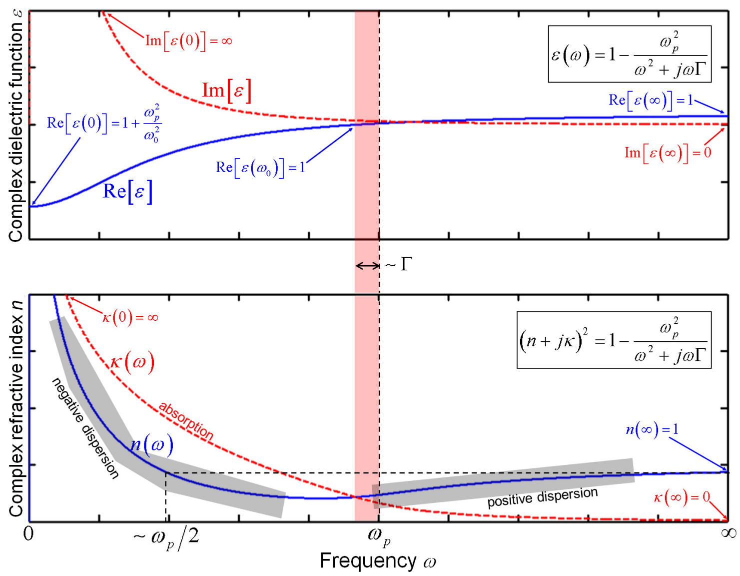
$$\nabla \times \vec{H} = j\omega\varepsilon_0\tilde{\varepsilon}_r\vec{E} = \sigma\vec{E} + j\omega\varepsilon_0\varepsilon_r\vec{E} \qquad \tilde{\varepsilon}_r = \varepsilon_r - j\frac{\sigma}{\omega\varepsilon_0}$$

Substituting the Drude equation into this result leads to expressions for the conductivity and the real-valued permittivity.

$$\varepsilon_r = 1 - \frac{\omega_p^2\tau^2}{1+\omega^2\tau^2} \qquad \sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \sigma_0 = \varepsilon_0\omega_p^2\tau$$

$\sigma_0 \equiv$ DC conductivity

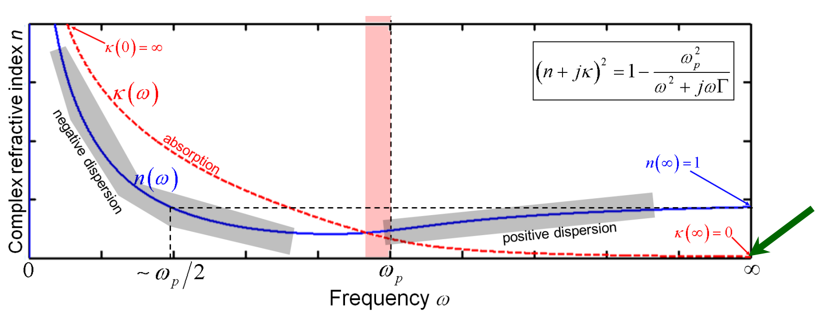
Typical Drude Response



Observation #1

At very high frequencies above the plasma frequency, loss vanishes and metals become transparent!

Note: more accurately stated as weakly absorbing



This is why we use x-rays to image through things.

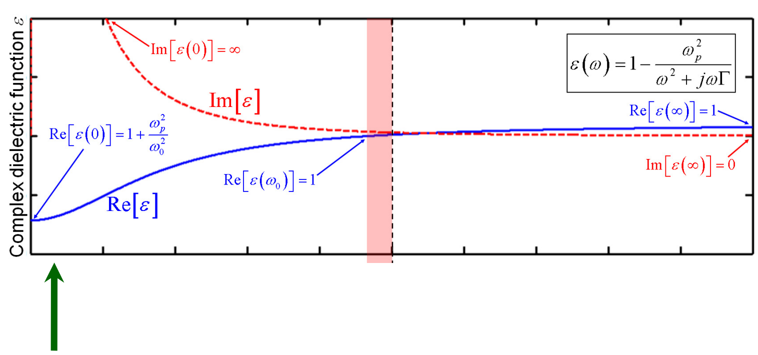
Observation #2

The plasma frequency for typical metals lies in the ultra-violet.

Metal	Symbol	Plasma Wavelength	Plasma Frequency
Aluminum	Al	82.78 nm	3624 THz
Chromium	Cr	115.35 nm	2601 THz
Copper	Cu	114.50 nm	2620 THz
Gold	Au	137.32 nm	2185 THz
Nickel	Ni	77.89 nm	3852 THz
Silver	Ag	137.62 nm	2180 THz

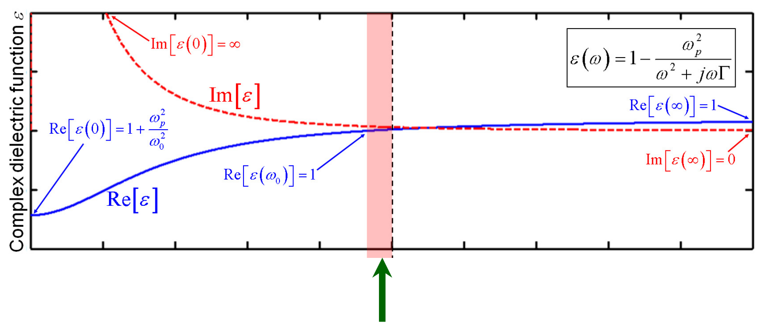
Observation #3

Below the plasma frequency, the dielectric constant is mostly imaginary and metals behave like good conductors.



Observation #4

Near the plasma frequency, both the real and imaginary parts of permittivity are significant and metals are **very lossy**.



This is a big problem for optics and currently the #1 limitation for optical metamaterials.

Low Frequency Properties of Metals

Most applications use frequencies well below ultraviolet so the behavior in this region is of particular interest.

For very low frequencies, $\omega \ll \frac{1}{\tau} \ll \omega_p$ the Drude model reduces to…

$$\tilde{\varepsilon}_r \cong 1 + j\frac{\sigma_0}{\omega} \qquad \begin{matrix} \varepsilon_r \cong 1 \\ \sigma \cong \sigma_0 \end{matrix}$$

The complex refractive index is then

$$\tilde{n} \cong (1+j)\sqrt{\frac{\sigma_0}{2\omega}} \quad \longrightarrow \quad \alpha(\omega) \cong \sqrt{\frac{\sigma_0 \omega}{2c_0^2}}$$

Skin Depth (at Low Frequencies)

Now that we know the complex refractive index, we can see how quickly a wave will attenuate due to the loss.

Skin depth is defined as the distance a wave travels where its amplitude decays by 1/*e* from this starting amplitude. This is simply the reciprocal of the absorption coefficient.

$$d(\omega) = \frac{1}{\alpha(\omega)} \cong \sqrt{\frac{2c_0^2}{\sigma_0 \omega}}$$

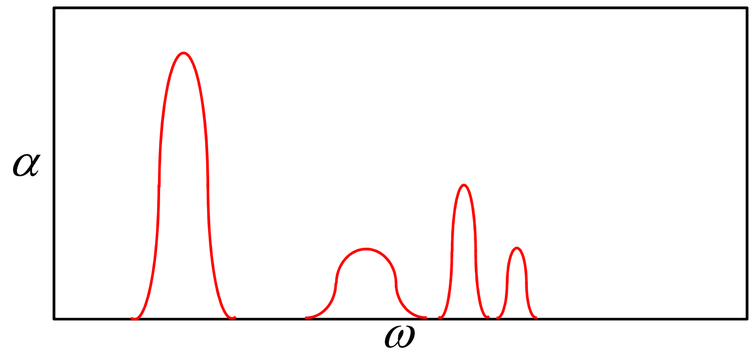
We see that higher frequencies experience greater loss and decay faster. For this reason, metallic structures are perform better at lower frequencies.

Generalizations

Real Atoms

Real atoms have up to tens of electron levels so there are many possibilities for electron resonances.

In addition, there are many sources of resonances other than electron transitions. (Spectrometry)

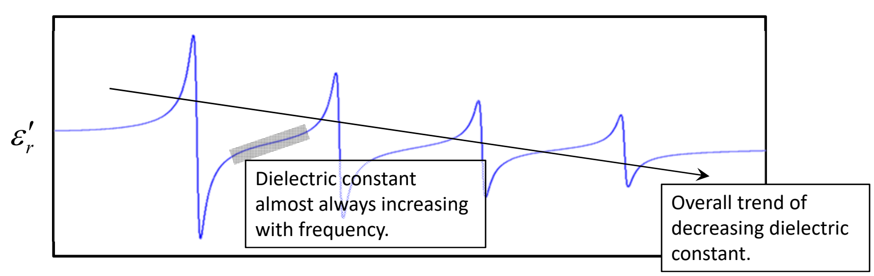


Accounting for Multiple Resonances

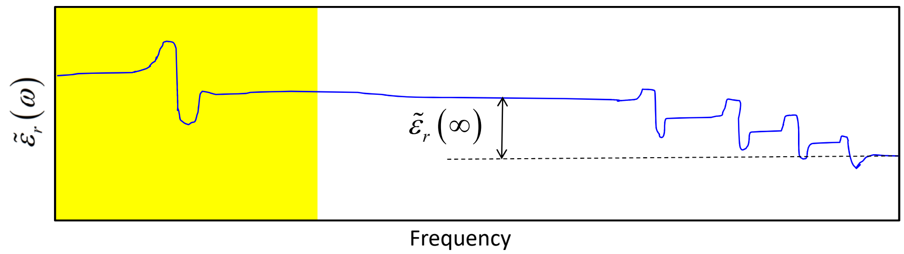
At a macroscopic level, all resonance mechanisms can be characterized using the Lorentz model. This allows any number of resonances to be accounted for through a simple summation.

$$\chi(\omega) = \omega_p^2 \sum_{i=1}^{N} \frac{f_i}{\omega_{0,i}^2 - \omega^2 - j\omega\Gamma_i}$$

$N \equiv$ Number of resonators
$f_i \equiv$ Oscillator strength of the i^{th} resonator
$\omega_{0,i} \equiv$ Natural frequency of the i^{th} resonator
$\Gamma_i \equiv$ Damping rate of the i^{th} resonator



$\varepsilon_r(\infty)$



Suppose we are only interested in obtaining $\varepsilon_r(\omega)$ within this span of frequencies.

In this frequency range, the high frequency resonances only contribute a combined DC offset that we write as $\varepsilon_r(\infty)$.

It is not necessary to resolve their Lorentz shape in this case and so we account for them solely through $\varepsilon_r(\infty)$.

Generalized Lorentz-Drude Model of Arbitrary Order

A very general equation for modeling complicated dielectrics and metals is the following:

$$\tilde{\varepsilon}_r(\omega) = \tilde{\varepsilon}_r(\infty) + \omega_p^2 \sum_{m=1}^{M} \frac{f_m}{\omega_{0,m}^2 - \omega^2 + j\omega\Gamma_m}$$

This is used to account for the offset produced by resonances at frequencies higher than where you care about.

LORENTZ-DRUDE PARAMETERS (eV)											
Parameter	Ag	Au	Cu	Al	Be	Cr	Ni	Pd	Pt	Ti	W
wp	9.010	9.030	10.830	14.980	18.510	10.750	15.920	9.720	9.590	7.290	13.220
f0	0.845	0.760	0.575	0.523	0.084	0.168	0.096	0.330	0.333	0.148	0.206
G0	0.048	0.053	0.030	0.047	0.035	0.047	0.048	0.008	0.080	0.082	0.064
w0	0	0	0	0	0	0	0	0	0	0	0
f1	0.065	0.024	0.061	0.227	0.031	0.151	0.100	0.649	0.191	0.899	0.054
G1	3.886	0.241	0.378	0.333	1.664	3.175	4.511	2.950	0.517	2.276	0.530
w1	0.816	0.415	0.291	0.162	0.100	0.121	0.174	0.336	0.780	0.777	1.004
f2	0.124	0.010	0.104	0.050	0.140	0.150	0.135	0.121	0.659	0.393	0.166
G2	0.452	0.345	1.056	0.312	3.395	1.305	1.334	0.555	1.838	2.518	1.281
w2	4.481	0.830	2.957	1.544	1.032	0.543	0.582	0.501	1.314	1.545	1.917
f3	0.011	0.071	0.723	0.166	0.530	1.149	0.106	0.638	0.547	0.187	0.706
G3	0.065	0.870	3.213	1.351	4.454	2.676	2.178	4.621	3.668	1.663	3.332
w3	8.185	2.969	5.300	1.808	3.183	1.970	1.597	1.659	3.141	2.509	3.580
f4	0.840	0.601	0.638	0.030	0.130	0.825	0.729	0.453	3.576	0.001	2.590
G4	0.916	2.494	4.305	3.382	1.802	1.335	6.292	3.236	8.517	1.762	5.836
w4	9.083	4.304	11.180	3.473	4.604	8.775	6.089	5.715	9.249	19.430	7.498
f5	5.646	4.384									
G5	2.419	2.214									
w5	20.290	13.320									
Min n:	0.130	0.080	0.100	0.033	1.350	0.310	0.860	0.720	0.730	0.760	0.490

The first resonance is not actually a resonance. Setting ω_0=0 defaults to the Drude model.

Isolated Absorbers in a Transparent Host

The overall material polarization is a superposition of the host and the absorber.

$$\vec{P}_{\text{total}} = \vec{P}_{\text{host}} + \vec{P}_{\text{absorber}}$$

The overall dielectric function is then

$$\tilde{\varepsilon}_r = 1 + \chi_{\text{host}} + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - j\omega\Gamma}$$

At very high frequencies relative to the absorber, this becomes

$$\tilde{\varepsilon}_r(\infty) = 1 + \chi_{\text{host}}$$

At very low frequencies relative to the absorber, this becomes

$$\tilde{\varepsilon}_r(0) = \tilde{\varepsilon}_r(\infty) + \frac{\omega_p^2}{\omega_0^2}$$

$$\boxed{\tilde{\varepsilon}_r(0) - \tilde{\varepsilon}_r(\infty) = \frac{\omega_p^2}{\omega_0^2}}$$

This provides a neat way to measure the plasma frequency.

Other Material Models

Cole-Cole Models

Cole-Cole models are physics-based compact representations of wideband frequency-dependent dielectric properties or polymers and organic materials.

$$\tilde{\varepsilon}(\omega) = \underbrace{\varepsilon_\infty + \frac{\sigma}{j\omega\varepsilon_0}}_{\text{DC or average response}} + \underbrace{\frac{\Delta\varepsilon}{1+(j\omega\tau)^{(1-\alpha)}}}_{\text{Dispersive response}}$$

$\varepsilon_\infty \geq 1$ $\quad\alpha \sim 0.1$

$\sigma \geq 0$

$\Delta\varepsilon \geq 0$

$\tau \geq 0$

α is an empirical parameter that accounts for the observed broad distribution of relaxation time constants.

K. S. Cole, R. H. Cole, "Dispersion and Absorption in Dielectrics I. Alternating Current Characteristics," J. of Chem. Phys. **9**, 341 (1941).

Cauchy Equation

This is an empirical relationship between refractive index and wavelength for transparent media at optical frequencies.

$$n(\lambda_0) = B + \frac{C}{\lambda_0^2} + \frac{D}{\lambda_0^4} + \cdots$$

$\lambda_0 \equiv$ free space wavelength in micrometers (μm)

B, C, D, etc. are called Cauchy coefficients.

For most materials, only B and C are needed.

$$n(\lambda_0) = B + \frac{C}{\lambda_0^2}$$

Material	B	C (μm^2)
Fused silica	1.4580	0.00354
Borosilicate glass BK7	1.5046	0.00420
Hard crown glass K5	1.5220	0.00459
Barium crown glass BaK4	1.5690	0.00531
Barium flint glass BaF10	1.6700	0.00743
Dense flint glass SF10	1.7280	0.01342

Sellmeier Equation

This is an empirical relationship between refractive index and wavelength for transparent media at optical frequencies.

$$n^2\left(\lambda_0\right)=1+\frac{B_1\lambda_0^2}{\lambda_0^2-C_1}+\frac{B_2\lambda_0^2}{\lambda_0^2-C_2}+\frac{B_3\lambda_0^2}{\lambda_0^2-C_3}$$

$\lambda_0 \equiv$ free space wavelength in micrometers (μm)

B_1, B_2, B_3, C_1, C_2, and C_3 are called Sellmeier coefficients.

Each term represents the contribution of a different resonance to refractive index. B_i is the strength of the resonance while $\sqrt{C_i}$ is the wavelength of the resonance in micrometers.

Crown Glass (BK7)

Coefficient	Value
B_1	1.03961212
B_2	0.231792344
B_3	1.01046945
C_1	6.00069867×10^{-3}
C_2	2.00179144×10^{-2}
C_3	1.03560653×10^{2}

The Sellmeier exists in other forms to account for additional physics.

$$n^2\left(\lambda_0\right)=A+\sum_i\frac{B_i\lambda_0^2}{\lambda_0^2-C_i} \qquad A\equiv n_\infty^2$$

There are other forms that account for temperature, pressure, and other parameters.