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Testing Stability by Quantifier Elimination

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For initial and initial-boundary value problems described by differential equations, stability requires the solutions to behave well for large times. For linear constant-coefficient problems, Fourier and Laplace transforms are used to convert stability problems to questions about roots of polynomials. Many of these questions can be viewed, in a natural way, as quantifier-elimination problems. The Seidenberg-Tarski theorem shows that quantifier-elimination problems are solvable in a finite number of steps. However, the complexity of this algorithm makes it impractical for even the simplest problems. The newer Quantifier Elimination by Partial Algebraic Decomposition (QEPCAD) algorithm is far more practical, allowing the solution of some non-trivial problems. In this paper, we show how to write all common stability problems as quantifier-elimination problems, and develop a set of computer-algebra tools that allows us to find analytic solutions to simple stability problems in a few seconds, and to solve some interesting problems in from a few minutes to a few hours.

1. Introduction

Initial-value problems for systems of ordinary differential equations, and initial-boundary value problems for systems of partial differential equations are one of the most common mathematical structures used to model physical processes. In general, these problems are nonlinear, and the first question asked about such problems is: is the problem well posed? That is, does the problem posses a unique solution that depends continuously on the data of the problem? If the problem is well posed, then the next most important questions concern the stability of the solutions. If two solutions that start out close together remain close together, then the problem is stable, while if the solutions become arbitrarily close, then the problem is asymptotically stable. Otherwise, the problem is unstable.

The analysis of such stability problems begins by studying system of equations that are linear and have constant coefficients. A standard approach to solving the linear constant-coefficient problems is to use transforms to reduce the problem to an algebraic form; in particular, if a variable is defined for all real values, the Fourier transform in that variable

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is used, while if the variable is defined for positive real values, the Laplace transform in that variable is used. The transforms and some algebra reduce the computation of solutions to the problem to finding the roots of a polynomial, called the *characteristic* polynomial, which is a polynomial in the transformed time variable with coefficients that are polynomials in the parameters of the problem and other transform variables.

The method of choice for solving a initial-boundary value problems for which analytic solutions cannot be found by the transform methods, is to approximate the problem by a discrete problem and then solve the discrete problem numerically. As in the continuum, it is important to know that the discrete problem is well posed, and if it is well posed, is it stable or asymptotically stable. In fact, more is required, since the discrete problems will contain discretization parameters: step sizes that go to zero, or the number of grid points that goes to infinity. The analysis of discrete problems follows the same pattern as the continuum problem. In particular, linear constant coefficient problems are analyzed first, using the same transforms as in the continuum problems.

For linear constant-coefficient problems, the notion of stability can be simplified a bit. A problem is well posed if all solutions are bounded in time by an exponential function, is stable if all solutions are bounded, and is asymptotically stable if all solutions converge to zero. The transform method of analysis in not particularly well suited to symbolic computing. However, the transform method shows that all solutions can be found by calculating all solutions of exponential form. Functions of the apropriate exponential form are called *trial solutions*. For example, for systems of ordinary differential equations, the *trial solutions* are a vector times an exponential function of the time variable t,

$$e^{\lambda t} \,,$$

where λ is the complex transform variable related to the time variable t. For this exponential to be a solution, λ will have to be a root of the characteristic polynomial, where typically the coefficients of the polynomial contain both parameters of the problem and other transform variables. Note that, when $t \to \infty$, if $\Re\lambda > 0$, the trial solution grows in time, while if $\Re\lambda = 0$, the trial solution is bounded, and if $\Re\lambda < 0$, the trial solution converges to zero. (The notation used in this paper is summarized in Appendix A.)

In the case of discrete problems, in particular for systems of ordinary difference equations, the *trial solutions* are also a vector times an exponential function of n, given by setting $t = n\,\Delta t$ in the previous trial solution. For computational purposes it is better to rewrite this function as a power:

$$e^{\lambda\, n\, \Delta t} = s^n \,, \quad s = e^{\lambda\, \Delta t} .$$

Again, note that when $n \to \infty$, if $|s| > 1$, the trial solution grows in time, while if $|s| = 1$, the trial solution is bounded, and if $|s| < 1$, the trial solution converges to zero.

Because of the way that exponentials grow, and given that the stability conditions must be uniform in the transform variables, one is tempted to say that a problem is well posed, if for all roots λ of the characteristic equation, there is a constant K (independent of the other transform variables) such that $\Re\lambda \leq K$, that the problem is stable if $\Re\lambda \leq 0$, and asymptotically stable if $\Re\lambda < 0$. The first and last of these conditions are ok, but the condition for stability is only ok if the polynomial has no multiple roots with $\Re\lambda = 0$. In the case of multiple roots, the solutions of the problem are exponentials, possibly multiplied by polynomials, and then the analysis becomes more difficult. Therefore, for problems in this paper, we restrict our attention to showing that problems are asymptotically stable or unstable. Similar comments apply to discrete problems.

How are such problems cast as quantifier-elimination problems? Asymptotic stability of the continuum problem for partial differential equations requires that all the roots of the characteristic polynomial have negative real part for all transform variables related to space. If the characteristic polynomial is $C(\lambda, \vec{\xi}, \vec{\alpha})$ where λ is the complex transform variable related to the time variable, the components of $\vec{\xi}$ are the real transform variables related to the spatial variables, and $\vec{\alpha}$ represents the parameters of the problem, then the asymptotic stability of the problem can be written

$$\forall \vec{\xi} \in \mathcal{R}\ \forall \lambda \in \mathcal{C}\ \{C(\lambda, \vec{\xi}, \vec{\alpha}) = 0 \Rightarrow \Re\lambda < 0\}\,. \tag{1.1}$$

A necessary condition for stability is obtained by replacing $<$ by $\leq$ in the last formula. The well-posedness can be stated as a similar quantifier elimination problem including existence of the global upper bound for the real part of the roots. In this paper we consider only asymptotic stability however well-posedness can be analyzed by similar methods. Eliminating the quantifiers from the previous logical formula produces a formula equivalent to the original but only involving the parameters $\vec{\alpha}$. This formula gives a complete answer to the stability problem.

The Tarski-Seidenberg Theorem says that the quantifiers in the previous logical formula can be eliminated in a finite number of steps to produce a logical formula in polynomials in $\vec{\alpha}$. However, all of the algorithms for eliminating quantifiers are extremely complex. In the literature on differential equations and numerical methods, stability problems are solved using variety of methods, but the solutions of even relatively modest problems are quite complex (e.g. Example 11.4.1, page 260, of Strikwerda (1989)). For difficult stability problems, it is typical to either prove an estimate of the stability condition (e.g. Wendroff (1991)), or to estimate the stability region using numerical sampling (e.g. Ganzha and Vorozhtsov (1987)). Fortunately, the quantifier-elimination problems arising in stability theory are neither completely general nor trivial. In the next section, a number of special techniques are described for analyzing and sometimes solving these problems. Because of the importance and complexity of stability problems, it is valuable to have a battery of tools for solving them, either approximately or exactly.

The solution procedures described in this paper start by using trial solutions to determine a characteristic polynomial for a problem and then writing the stability problem as a quantifier-elimination problem. If the problem concerns asymptotic stability of an initial-value problem (see Equation (1.1)), then a Routh-Hurwitz type procedure is used to eliminate one complex quantified variable (λ). For ordinary differential or difference equations, this along with some simplification, solves the problem. For partial differential or difference equations (for these a conformal map is applied before Routh-Hurwitz), the remaining quantifiers are eliminated using a *quantifier elimination using partial cylindrical algebraic decomposition* algorithm (QEPCAD). For stability problems (Equation (1.1) with $<$ replaced by $\leq$), the Routh-Hurwitz type criteria do not apply, so QEPCAD must be applied directly, and this makes such problems much more difficult. The quantifier elimination problems for the stability of boundary conditions are significantly different from those for initial-value problems.

The ideas about using trial solutions have been known for centuries while the ideas about stability and the use of the Routh-Hurwitz theorem go back to the previous century. Recall that we are dealing only with the simplest assymptotic stability, while for numerically solving ODEs there exist a deep stability theory (Hairer, Norsett and Wanner, 1993; Dahlquist, 1985) dealing with other stability concepts. One thing that is new in this paper is the recognition that all common well-posedness and stability questions typically asked about continuum and discrete initial boundary-value problems can be stated as quantifier elimination problems. In particular, it was not evident that the stability of boundary conditions could be written this way. Moreover, it is shown that QEPCAD augmented by Routh-Hurwitz type procedures, provides a tool that will solve most textbook examples, some more complex examples, and a few research examples. In particular, the results for the LeVeque scheme given in Example 5.2.2 are new. We believe that because improvement of quantifier-elimination algorithms is now a very active area of research, far more complex examples will be tractable in the near future. To make this paper readable for non-experts in stability theory, we have developed a consistent notation and terminology for stability problems and we have included many elementary examples. The last two authors have previously discussed the stability of discretized initial value PDE problems by quantifier elimination in Liska and Steinberg (1993).

A detailed discussion of the quantifier elimination techniques used in this paper are presented in the next section (Section 2), and in Section 3 a more mathematical statement of the stability problems is given. In Section 4, the solutions of some elementary problems for systems of ordinary differential and difference equations are presented. These problems are easily solved but, in the case of problems for ODEs from control theory, there are several nontrivial problem discussed in Abdallah, Dorato, Yang, Liska and Steinberg (1996). In Section 5, initial value problems for systems of both partial differential and difference equations are discussed. This quickly leads to some nontrivial problems for some commonly used discretizations. In the final section (Section 6), initial-boundary value problems for partial differential and semi-discrete equations are discussed. Here, even simple problems are too complex to be solved easily, and more work needs to be done to be able to tackle hard problems.

2. Mathematical Tools

In this section, the mathematical and computer-algebra tools, which are regularly used for stability analysis, are described. The most useful tools are various versions of the Routh-Hurwitz Criterion that are well known in stability theory and the partial cylindrical algebraic decomposition algorithm. The Routh-Hurwitz type criteria only remove one quantified variable, and many problems encountered in our applications have additional quantified variables, so such criteria are not sufficient to complete the task, so this is backed up by a powerful set of quantifier-elimination algorithms. Another important tool is a conformal map that is used to convert problems about discrete equations into the same form as continuum problems. The notation used is summarized in Appendix A.

2.1. Routh-Hurwitz and Related Criteria

In the stability analysis of continuum problems, a typical quantifier-elimination problems has the form

$$\forall r \in \mathcal{C}\ \{P(r, \vec{a}) = 0 \Rightarrow \Re r < 0\}\,, \tag{2.1}$$

where P is a polynomial in the variable r and the parameters $\vec{a}$ which are the coefficients of the polynomial P. This is the problem of showing that all of the roots of the polynomial P are in the left half of the complex plane.

The Routh-Hurwitz problem is to decide if all roots of a polynomial have negative real part or not. For real numerical coefficients, the Routh-Hurwitz Criterion (see Gantmacher

(1959)) gives a decision procedure for this problem. This criterion contains conditions on the coefficients of the polynomial and the Hurwitz determinants, which are minors of the Hurwitz matrix constructed from these coefficients. A polynomial, all roots of which have negative real part, is called a *Hurwitz* polynomial. For symbolic real coefficients, which is our principal interest, the Routh-Hurwitz Criterion produces a logical formula involving polynomial inequalities in the coefficients of the polynomial which is equivalent to (2.1).

Simpler formulas are obtained from the Liénard-Chipart Criterion (see Gantmacher (1959)), which is what we typically use in our analyzes. For complex coefficients, the generalized Routh-Hurwitz Criterion, which produces an equivalent formula using polynomial inequalities involving the real and imaginary parts of the coefficients, solves the quantifier-elimination problem (2.1).

2.2. Conformal Map

For the stability analysis of discrete problems, a typical quantifier-elimination problem has the form

$$\forall r \in \mathcal{C} \; \{P(r, \vec{a}) = 0 \Rightarrow |r| < 1\} \,. \tag{2.2}$$

The well-known conformal map

$$r = \frac{z+1}{z-1}\,, \quad z = \frac{r+1}{r-1}\,,$$

which was previously used by Ganzha and Voroztsov (1987) in the same context, gives a bijection between the interior of the unit circle and the interior of the left-half plane. If $P(r) = P(r, \vec{a})$ and

$$Q(z) = (z-1)^N P\left(\frac{z+1}{z-1}\right)\,, \tag{2.3}$$

where N is the degree of $P(r)$, then

$$\{P(r) = 0 \Rightarrow |r| < 1\} \Leftrightarrow \{Q(z) = 0 \Rightarrow \Re z < 0\}\,,$$

so the linear-fractional map produces an equivalence between quantifier-elimination problems of the form (2.2) and (2.1). Linear fractional maps between the unit circle and a half plane were introduced by Cayley (see the comment on page 237 of Hille 1962) and later found important applications in the functional analysis of unbounded operators (see page 202 in Yosida 1966).

2.3. Quantifier Elimination Tools

Around 1930, Tarski (1930) proved that quantifier elimination (for the first order theory of the real numbers) is possible. This was one of the most fundamental results in mathematical logic (along with Goedel's proof of incompleteness). In 1951, with the assistance of McKinsey (at RAND cooperation), he gave a rigorous and algorithmic description of the proof, in the hope that the algorithm might be programmed on a computer. But the complexity of the algorithm was prohibitive: the running time could not be bound by any finite tower of exponential functions. Later somewhat simpler methods were devised by Seidenberg (1954) and Cohen (1969). But the complexity remained the same, and the early hope for developing computer tools for mechanical quantifier elimination waned.

In 1975, Collins made a breakthrough by providing a completely new method whose time complexity was only doubly exponential, thus reducing the height of the tower of exponential functions from infinity to just two! The method was based a construction that he called "cylindrical algebraic decomposition (CAD)".

It renewed a hope for mechanical quantifier elimination and inspired many further improvements: clustering (Arnon (1981), Arnon(1988)), adjacency (Arnon et al. (1984), Arnon et al. (1988), Collins and McCallum (1995)), reduced projection (McCallum (1984), McCallum (1988)), strict inequalities (McCallum (1993)), another reduced projection (Hong (1990a)), partial CAD (Hong (1990b), Collins and Hong (1991), Hong (1993a)), simple solution formula (Hong (1990b), Hong (1992)), efficient root isolation of real algebraic polynomials (Johnson (1991), Johnson (1992)), efficient computation of greatest common divisor of real algebraic polynomials (Langemyr and McCallum (1989), Encarnación (1994)), equational constraints (Collins (1994)), generic projection and lifting (Hong (1995b)), simple solution formula (Brown and Collins (1996)), parallelism (Saunders et al. (1990), Hong (1993b)), etc. In our work, we used the program QEPCAD (Quantifier Elimination by Partial Cylindrical Algebraic Decomposition) which was developed by Hong (1990b) with contributions by Brown, Collins, Encarnacion, and Johnson. This program incorporates most of the improvements listed above.

During the last several years, a few new general methods (Grigor'ev (1988), Canny (1993), Heintz et al. (1989), Renegar (1992)) were proposed. Their time complexity is singly exponential when all the quantifiers are the same. As far as we aware, there is no implementation of these methods. As soon as their implementations are available, it will be interesting to test their applicability to stability problems.

Although the quantifier-elimination problems can be quite complex, they sometimes have many special features that allow special algorithms to be applied. For example, in Abdallah et al. (1996) the algorithms developed in Weispfenning (1988) and in Loos and Weispfenning (1993) for linear problems were used to solve some problems in control theory. For quadratic problems, one can use the algorithms of Weispfenning (1993) and Hong (1993c); and for cubic problems, one can use Weispfenning (1994a). These algorithms are able to treat problems where the quantified variables have low degree, while the non-quantified variables can have arbitrary degree. The preliminary implementation of the linear and quadratic cases shows that, as expected, these algorithms are much faster than the QEPCAD algorithm, and can treat many more variables. Unfortunately, these algorithms typically produce very large un-simplified answers. However, the simplification techniques being developed by Dolzmann and Sturm (1996) might help.

Another new practical approach to several special quantifier-elimination problems is based on counting real zeros using Sturm-Habicht sequences, see González-Vega (1994) and González-Vega, Lombardi, Recio and Roy (1994), and was applied to well-posedness of continuum initial-value problems for partial differential equations in Liska and Wendroff (1996).

Still another approach is based on the observation that it is often sufficient to get some approximation to the answers. For instance, a good estimate of a largest box contained in the solution set is sufficient for many applications. In Hong (1995a) and Hong and Neubacher (1996), a method based on space decomposition and tightening (Hong and Stahl (1994)) is given. When approximation is acceptable, this method is promising.

		ODE IVP	PDE IVP	PDE IBVP
Continuum				
	System	$\mathbf{S}(\partial_t,\, \vec{\alpha})$	$\mathbf{S}(\partial_t,\, \partial_{\vec{x}},\, \vec{\alpha})$	$\mathbf{S}(\partial_t,\, \partial_y,\, \partial_{\vec{x}},\, \vec{\alpha})$
	Trial	$e^{\lambda\, t}$	$e^{\lambda\, t+\imath\, \vec{\xi}\, \vec{x}}$	$e^{\lambda\, t+\eta y+\imath\, \vec{\xi}\, \vec{x}}$
Discrete				
	System	$\mathbf{S}(\delta_t,\, \vec{\alpha})$	$\mathbf{S}(\delta_t,\, \delta_{\vec{x}},\, \vec{\alpha})$	$\mathbf{S}(\delta_t,\, \delta_y,\, \delta_{\vec{x}},\, \vec{\alpha})$
	Trial	s^n	$s^n\, \vec{z}^{\vec{j}}$	$s^n\, w^m\, \vec{z}^{\vec{j}}$

Table 1. The Choice of Trial Functions

3. Stability Problems

We study stability problems for initial-value problems for ordinary-differential and partial differential equations, and initial-boundary value problems for partial differential equations, and also discrete analogs of these problems. All of these problems can be written in a common form:

$$\mathbf{S}\,\vec{f} = 0 \tag{3.1}$$

where $\mathbf{S}$ is an $N \times N$ matrix of differential or difference operators and $\vec{f}$ is a N vector of unknown functions. These equations must be supplemented by appropriate initial conditions, but these do not play a role in the analysis. In the case of a boundary-value problem, the boundary conditions are written

$$\mathbf{B}\,\vec{f} = 0\,, \tag{3.2}$$

where $\mathbf{B}$ is an $M \times N$ matrix of differential or difference operators. In all cases, the problems are linear and constant coefficient, and depend on some set of parameters $\vec{\alpha}$, so

$$\mathbf{S} = \mathbf{S}(\vec{D},\, \vec{\alpha})\,, \tag{3.3}$$

where $\vec{D} = (D_1, D_2, \cdots, D_k)$ and the D_i are either a differential or difference operators. The boundary conditions have exactly the same dependencies. The critical assumption is:

Assumption: Both $\mathbf{S}$ and $\mathbf{B}$ are polynomials of both of their arguments $\vec{D}$ and $\vec{\alpha}$.

The stability is checked by computing special solutions, well known already to Euler and Lagrange, of the given problem, which are products of a trial solution and a constant vector. The trial functions depend on the problem being analyzed and are summarized in Table 1, along with the dependencies of $\mathbf{S}$. The connection between the trial solutions for the continuum and discrete problems is given by

$$s = e^{\lambda\, \Delta t}\,, \quad z_k = e^{\imath\, \xi_k\, \Delta x_k}\,, \quad w = e^{\eta\, \Delta y}\,. \tag{3.4}$$

The connection between the trial solutions and stability is given by noting how the trial solutions grow: in the continuum case, as t, y, and $\vec{x}$ go to infinity; and in the discrete case, as n, m, and $\vec{j}$ go to infinity. In all cases, it is assumed the $\vec{\xi}$ is a real vector and that $|z_k| = 1$. So the trial solutions are bounded as functions of $\vec{x}$ are $\vec{j}$. If $\Re\lambda < 0$ or $|s| < 1$, then the trial solutions converge to zero as t or n go to infinity, while if $\Re\eta < 0$ or $|w| < 1$, the trial solutions converge to zero as y or m go to infinity.

In fact, we do not analyze fully discretized initial-boundary value problems, but instead consider semi-discrete problems where time is not discretized:

$$\mathbf{S} = \mathbf{S}(\partial_t, \delta_y, \delta\vec{x}, \vec{\alpha}) \tag{3.5}$$

and then the trial solutions have the form

$$\vec{f} = e^{\lambda t}\, w^m\, \vec{z}^{\vec{j}}\, \vec{v}(\lambda, w, z)\,. \tag{3.6}$$

Substituting the trial solution into (3.1) and then dividing by the trial solution gives

$$\mathbf{S}(\lambda,\, w,\, z,\, \vec{\alpha})\,\vec{v} = 0 \tag{3.7}$$

This can have a nontrivial solution if and only if

$$C(\lambda,\, w,\, z,\, \vec{\alpha}) = \det \mathbf{S}(\lambda,\, w,\, z,\, \vec{\alpha}) = 0\,, \tag{3.8}$$

where C is called the *characteristic* polynomial and is a polynomial in all of its arguments.

The analysis of all initial-value problems proceeds in the same way, only the variables change. The analysis of boundary conditions is quite different and is discussed later. We only consider cases for which the characteristic polynomial has simple roots. The multiple root case is important, far more difficult, and in progress.

4. Ordinary Differential Equations

We begin with some simple examples for ordinary differential and difference equations, and finish with the interesting example of the local stability of Runge-Kutta methods.

4.1. Continuous Equations

In case case of ordinary differential equations, $\mathbf{S} = \mathbf{S}\,(\partial_t,\, \vec{\alpha})$, and the trial solutions have the form

$$\vec{f}(t) = e^{\lambda t}\,\vec{v}(\lambda)\,, \tag{4.1}$$

where $\lambda \in \mathcal{C}$. The characteristic polynomial is $C(\lambda,\, \vec{\alpha}) = \det\, \mathbf{S}(\lambda,\, \vec{\alpha})$. For an extensive discussion of the stability of ODEs, see e.g. Braun (1978) p. 354.

A system of ODEs is *asymptotically stable* if and only if all solutions of the initial-value problem converge to zero as $t \to \infty$, and this is equivalent to the quantifier-elimination problem

$$\forall\lambda \in \mathcal{C}\ \{C(\lambda,\, \vec{\alpha}) = 0 \Rightarrow \Re\,\lambda < 0\}\,. \tag{4.2}$$

The solution of this QE problem gives a region in the parameter space $\vec{\alpha}$ for which the initial-value problem is asymptotically stable. This QE problem is a Routh-Hurwitz problem, and thus can be solved by methods described in Section 2.1.

There are many problems related to the stability of ODEs in control theory that can be reduced to QE problems quite different from (4.2). Several of these are described

in Abdallah, Dorato, Yang, Liska and Steinberg (1996) or Dorato, Yang and Abdallah (1996).
Example 4.1: The single second-order ODE,

$$a\,\frac{d^2 f}{d\,t^2} + b\,\frac{d\,f}{d\,t} + c\,f = 0\,, \tag{4.3}$$

with parameters $\vec{\alpha} = (a, b, c)$, $a \neq 0$, provides a very simple example. The characteristic polynomial is

$$C(\lambda,\, a,\, b,\, c) = a\,\lambda^2 + b\,\lambda + c\,. \tag{4.4}$$

and the ODE (4.3) is asymptotically stable if and only if

$$\forall \lambda \in \mathcal{C}\ \{a\,\lambda^2 + b\,\lambda + c = 0 \Rightarrow \Re\,\lambda < 0\}\,.$$

This QE problem can be solved easily by hand or by using the Routh-Hurwitz method, which shows that the asymptotic stability of (4.4) is equivalent to $(a > 0 \wedge b > 0 \wedge c > 0) \vee (a < 0 \wedge b < 0 \wedge c < 0)$.

4.2. Discrete Equations

In the case of discrete ordinary difference equations, $\mathbf{S} = \mathbf{S}\,(\delta_t,\, \vec{\alpha})$, and the trial solutions have the form

$$\vec{f}^n\,(s) = s^n\,\vec{v}(s)\,, \tag{4.5}$$

where $s \in \mathcal{C}$. The characteristic polynomial is $C(s,\, \vec{\alpha}) = \det \mathbf{S}(s,\, \vec{\alpha})$. A complete discussion of stability for ordinary differential equations can be found in Hairer and Wanner (1996).

As with ODEs, a difference scheme is asymptotically stable if and only if all solutions of the initial value problem converge to zero as $n \to \infty$ and this is equivalent to the QE problem

$$\forall s \in \mathcal{C}\ \{C(s,\, \vec{\alpha}) = 0 \Rightarrow |s| < 1\}\,.$$

This QE problem is in the form (2.2), and thus can be transformed into a Routh-Hurwitz problem using the conformal map (2.3), and then solved using the methods describe in Section 2.1.
Example 4.2: The second-order ODEs (4.3) considered in Example 4.1 is equivalent to the system of ODE,

$$\frac{d\,g}{d\,t} = -\frac{b}{a}\,g - \frac{c}{a}\,f\,, \quad \frac{d\,f}{d\,t} = g\,.$$

This system can be discretized using a simple explicit scheme:

$$\frac{\delta - 1}{\Delta t}\,g^n = -\frac{b}{a}\,g^n - \frac{c}{a}\,f^n\,, \quad \frac{\delta - 1}{\Delta t}\,f^n = g^n\,,$$

and then the characteristic polynomial for the discrete system is

$$C(s) = \det \begin{pmatrix} \frac{s-1}{\Delta t} + \frac{b}{a} & \frac{c}{a} \\ -1 & \frac{s-1}{\Delta t} \end{pmatrix}.$$

The linear-fractional mapping technique followed by the Routh-Hurwitz Criterion shows that the discrete system is asymptotically stable for

$$c(b - c\Delta t) > 0 \wedge c(c\Delta t^2 - 2b\Delta t + 4a) > 0$$

which gives the limit on time step Δt.

4.3. Local Stability of Runge-Kutta Discretizations

Runge-Kutta (RK) methods are multi-stage methods for computing approximations $\vec{f}^{(n)}$ to $\vec{f}(n\,\Delta t)$, where several intermediate quantities $\vec{g}_k$ are computed and then used to approximate $\vec{f}^{n+1}$. The first-order system of ODEs has

$$\mathbf{S}\,(\partial_t,\,\vec{\alpha}) = \partial_t \mathbf{I} - \mathbf{A}(\vec{\alpha})$$

and then a m-stage scheme for approximating the initial-value problem is given in terms of a $m \times m$ real matrix $\mathbf{a}$ and a real row vector $\vec{b}$ of length m:

$$\begin{aligned} \vec{g}_k &= \vec{f}^n + \Delta t\,\mathbf{A}\sum_{j=1}^{m} a_{k,j}\,\vec{g}_j\,, \quad k = 1,\cdots,m \\ \vec{f}^{n+1} &= \vec{f}^{(n)} + \Delta t\,\mathbf{A}\sum_{j=1}^{m} b_j\,\vec{g}_j\,. \end{aligned} \tag{4.6}$$

See Wanner (1969) or Hairer and Wanner (1996) for more details on Runge-Kutta methods.

Because the system is first order, the vector $\vec{v}$ in the trial solution will be an eigenvector of $\mathbf{A}$ with eigenvalue μ. The characteristic polynomial is $C(s,\,\Delta t\,\mu)$, so if $z = \Delta t\,\mu$, then the *characteristic function* C is defined to be

$$C(s,\,z) = s - R(z)\,.$$

The *stability function* R is given by

$$R(z) = 1 + z\,\vec{b}\,(\mathbf{I} - z\,\mathbf{a})^{-1}\,\vec{1} = \frac{\det(\mathbf{I} - z\,\mathbf{a} + z\,\vec{1}\,\vec{b})}{\det(\mathbf{I} - z\,\mathbf{a})}\,,$$

where $\vec{1} = (1,\cdots,1)^T$ (see Hairer and Wanner (1996) for more details). For implicit RK methods, the stability function $R(z)$ is a rational function. An RK method is explicit if the matrix $\mathbf{a}$ is lower triangular with zero diagonal, $\mathbf{a}_{k,j} = 0$ for $j \geq k$, and in this case, R is a polynomial as is easily seen from the fact that the determinant of $(\mathbf{I} - z\,\mathbf{a})$ is one. The stability domain of a RK method is defined by

$$\{z \in \mathcal{C};\; C(s,z) = 0 \Rightarrow |s| < 1\}\,,$$

which can be rewritten as

$$\{z \in \mathcal{C};\; |R(z)| < 1\}\,. \tag{4.7}$$

Recently Kreiss and Scherer (1992) and Kreiss and Wu (1993) have pointed out the importance of the notion of *local stability*. The RK method (4.6) is locally stable if there exists $\rho > 0$ so that the stability domain (4.7) of the method contains a half circle

$$\{z \in \mathcal{C};\; \Re z < 0 \wedge |z| < \rho\}\,.$$

For explicit RK methods, Kreiss and Scherer (1992) give a simple method for checking the local stability. However, for practical use of the local stability condition, it is helpful to know the *maximal* ρ for which the half-circle is included in the stability region. For RK methods, the computation of the maximal ρ is a QE problem

$$\forall z \in \mathcal{C}\;\; \{\{\Re z < 0 \wedge |z| < \rho\} \Rightarrow |R(z)| < 1\}\,. \tag{4.8}$$

Example 4.3.1: Heun's (3-rd order Runge-Kutta) method has the stability function (see Hairer and Wanner (1996))

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}\,.$$

In this case, the quantifier-elimination problem for finding the maximal ρ for which the half-circle is in the stability region is to eliminate the quantifiers from (4.8) written in terms of $z = x + \imath\, y$:

$$\forall x \in \mathcal{R}\ \forall y \in \mathcal{R}\ \left\{ \{x < 0 \wedge x^2 + y^2 < \rho^2\} \Rightarrow R(x + \imath\, y)\, R(x - \imath\, y) < 1 \right\}\,.$$

Note that the quantifier-elimination problem only involves real polynomials in real variables. In 91 seconds, the QEPCAD package calculated the equivalent quantifier free formula $\rho^2 \leq 3$, which agrees with Kreiss and Scherer (1992) (there is a typo in appendix of Kreiss and Scherer (1992)).

Example 4.3.2: The stability function of the Dormand-Prince 5-th order embedded 7 stage method is (see Hairer and Wanner (1996))

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \frac{z^6}{600}\,.$$

Finding the stability region for this problem is too difficult for QEPCAD. However, not all is lost. From a plot of the stability region, it appears that the border of the stability domain (4.8) crosses the positive imaginary axis at one point ρ, where ρ^2 can be found by solving a cubic equation given by $\rho^6 - 25\,\rho^4 + 225\,\rho^2 - 200 = 0$. A numerical routine for finding roots estimates that ρ^2 is in the interval $(0.99438, 0.99439)$. In 941 seconds, QEPCAD can verify that for $\rho^2 = 0.99438$, statement (4.8) is true, and in 282 seconds, that for $\rho^2 = 0.99439$, statement (4.8) is false, proving that maximal ρ falls within this interval.

5. Initial Value Problems for Partial Differential Equations

From the quantifier-elimination point of view, initial-value problems for partial differential equations are more interesting because they contain more quantifiers, more precisely, one universal quantifier for each spatial variable in the PDE. Consequently, applying a Routh-Hurwitz type procedure produce a quantifier-elimination problem rather than solving the problem as in the ordinary differential equation case.

5.1. Continuous Equations

In the case of initial-value problems for partial differential equations, $t \geq 0$, $\vec{x} \in \mathcal{R}^d$ and $\mathbf{S} = \mathbf{S}(\partial_t, \partial_{\vec{x}}, \vec{\alpha})$. The trial solution has the form

$$\vec{f}(t, \vec{x}) = e^{\lambda\, t + \imath\, \vec{\xi} \cdot \vec{x}}\, \vec{v}(\lambda, \vec{\xi})\,, \quad \vec{\xi} = (\xi_1, \cdots, \xi_d)\,,$$

where $\lambda \in \mathcal{C}$ and $\vec{\xi} \in \mathcal{R}^d$. The characteristic polynomial is $C(\lambda, \vec{\xi}, \vec{\alpha}) = \det \mathbf{S}(\lambda, \imath\vec{\xi}, \vec{\alpha})$. For more detailed discussion see Chapter 2 of Kreiss and Lorenz (1989) or chapter 9 of Strikwerda (1989).

The system of PDEs is *asymptotically stable* if and only if all solutions that are bounded in $\vec{x}$, converge to zero as $t \to \infty$. This is equivalent to the QE problem

$$\forall \vec{\xi} \in \mathcal{R}^d\ \forall \lambda \in \mathcal{C}\ \{C(\lambda, \vec{\xi}, \vec{\alpha}) = 0 \Rightarrow \Re\, \lambda < 0\}\,, \tag{5.1}$$

that is, for all $\vec{\xi}$ the polynomial C is a Hurwitz polynomial in λ. This type of QE problem is solved by eliminating λ using one of the Routh-Hurwitz method described in Section 2.1, then simplifying the intermediate results, and then applying one of the QE methods given in Section 2.3.

Example 5.1.1 The easiest equations to analyze are those that are first order in time:

$$\frac{\partial f}{\partial t} + a\,\frac{\partial^4 f}{\partial x^4} + b\,\frac{\partial^3 f}{\partial x^3} + c\,\frac{\partial^2 f}{\partial x^2} + d\,\frac{\partial f}{\partial x} + g\,f = 0\,, \tag{5.2}$$

where $f = f(t,\,x), a, b, c, d, g \in \mathcal{R}$. The characteristic polynomial for Equation (5.2) is

$$C(\lambda,\,\xi) = \lambda + a\,\xi^4 - b\,\imath\,\xi^3 - c\,\xi^2 + d\,\imath\,\xi + g\,,$$

where the dependence of $C(\lambda,\,\xi)$ on the parameters has been suppressed.

The equation $C(\lambda,\,\xi) = 0$ can be solved for λ and this can be used to change the QE problem (5.1) to the simple QE problem

$$\forall \xi\ \{-a\,\xi^4 + c\,\xi^2 - g < 0\}\,.$$

Because this QE problem is quadratic in ξ^2, it can be solved by hand by finding the local extrema of $\Re\,\lambda$:

$$[a > 0 \wedge g > 0 \wedge (c \leq 0 \vee (c > 0 \wedge c^2 - 4\,g\,a < 0))] \vee [a = 0 \wedge g > 0 \wedge c \leq 0]\,.$$

QEPCAD produced an equivalent but simpler formula

$$(a \geq 0 \wedge c \leq 0 \wedge g > 0) \vee (4\,a\,g - c^2 > 0 \wedge g \geq 0)$$

in just 0.5 s CPU time.

Example 5.1.2 The system of differential equations

$$\begin{aligned}
\frac{\partial\,u}{\partial\,t} &= c\,\frac{\partial\,w}{\partial\,x} - (\gamma - 1)\,\frac{\partial\,e}{\partial\,x}\,,\\
\frac{\partial\,w}{\partial\,t} &= c\,\frac{\partial\,u}{\partial\,x}\,,\\
\frac{\partial\,e}{\partial\,t} &= \sigma\frac{\partial^2 e}{\partial\,x^2} - c\,\frac{\partial\,u}{\partial\,x}\,,
\end{aligned} \tag{5.3}$$

describe coupled sound and heat flow (see Richtmyer and Morton (1967)). Note that the constant functions are solutions of this problem, so the equations cannot be asymptotically stable. As the functions constant in space correspond to $\vec{\xi} = 0$, we will analyze the stability for $\xi \neq 0$.

The characteristic matrix for the system is

$$\mathbf{S}(\lambda,\,\xi) = \begin{pmatrix} \lambda & -\imath\,c\,\xi & \imath\,(\gamma - 1)\,\xi \\ -\imath\,c\,\xi & \lambda & 0 \\ \imath\,c\,\xi & 0 & \lambda + \sigma\,\xi^2 \end{pmatrix}\,,$$

and the characteristic polynomial is

$$C(\lambda,\,\xi) = \lambda^3 + \sigma\,\xi^2\,\lambda^2 + c\,(c + \gamma - 1)\,\xi^2\,\lambda + c^2\,\sigma\,\xi^4\,.$$

The Routh-Hurwitz Criterion is applied to the asymptotic stability condition (5.1) with $\xi \neq 0$ and the result is simplified to give equivalent condition

$$\sigma > 0 \wedge c\,(\gamma - 1) > 0\,.$$

This condition is true for $c > 0$, $\sigma > 0$ and $\gamma > 1$, which are all of the physically meaningful values of the constants that appear in (5.3).

Difference approximations to the system of PDEs (5.3) were investigated in Richtmyer and Morton (1967) and Liska and Steinberg (1993).

5.2. Discrete Equations

In the case of initial-value problems for partial difference equations, $\mathbf{S} = \mathbf{S}(\delta_t,\, \delta_{\vec{x}},\, \vec{\alpha})$, and the trial solution has the form

$$\vec{f}^{\,n}_{\vec{j}} = s^n\, \vec{z}^{\,\vec{j}}\, \vec{v}\,, \quad |z_k| = 1\,, \quad k = 1, \cdots d\,, \tag{5.4}$$

where $s \in \mathcal{C}$ and $\vec{z} \in \mathcal{R}^d$. The characteristic polynomial is $C(s,\, \vec{z},\, \vec{\alpha}) = \det \mathbf{S}(s,\, \vec{z},\, \vec{\alpha})$. In the literature, it is rather common to use exponential trial solutions rather than powers. The connection between the two notations is given by (3.4). The power form of the solution has the advantage that the characteristic function is a polynomial and not a polynomial in exponentials.

The difference scheme is asymptotically stable if and only if all solutions of the initial-value problem which are bounded in $\vec{z}$ converge to zero as $n \to \infty$. The asymptotic stability is guaranteed by the von Neumann sufficient condition for stability of the difference scheme:

$$\forall \vec{z} \in \mathcal{C}^d\ \forall s \in \mathcal{C}\ \ \{\{|\vec{z}| = 1 \wedge C(s,\, \vec{z},\, \vec{\alpha}) = 0\} \Rightarrow |s| < 1\}\ , \tag{5.5}$$

where $|\vec{z}| = 1$ means $|z_1| = 1 \wedge |z_2| = 1 \wedge \cdots \wedge |z_d| = 1$. The von Neumann necessary condition for stability is obtained by replacing $|s| < 1$ in (5.5) by $|s| \le 1$. Note that solutions which are unbounded in space are not used to study the stability properties of the initial value problem.

For some special cases, e.g. for a scalar one-step difference scheme ($N = 1$ and $\mathbf{S}_{1,1}$ linear in δ_t), the von Neumann necessary condition is also sufficient. For this case, the characteristic polynomial is linear in s, so the equation $C(s,\, \vec{z},\, \vec{\alpha}) = 0$ can solved for $s = g(\vec{z},\, \vec{\alpha})$, where g is called the *amplification factor*, and then the von Neumann necessary and sufficient condition for stability can be written as

$$\forall \vec{z} \in \mathcal{C}^d\ \{|\vec{z}| = 1 \Rightarrow |g(\vec{z},\, \vec{\alpha})| \le 1\}\ . \tag{5.6}$$

For a more detailed discussion of stability of difference schemes see e.g. Strikwerda (1989) or Richtmyer and Morton (1967). The use of QE for the stability analysis of difference schemes for initial-value problems for PDEs was presented in Liska and Steinberg (1993).

The von Neumann stability condition (5.5) for asymptotic stability is analyzed by first using the conformal map (2.3) to transform the problem into a Routh-Hurwitz problem, which is then analyzed by the Routh-Hurwitz methods described in the Section 2.1. In addition, the complex variables $\vec{z}$ are transformed to real variables using trigonometric functions:

$$z_k = \cos(\xi_k\, \Delta x_k) + \imath \sin(\xi_k\, \Delta x_k)\,, \quad k = 1, \cdots, d \tag{5.7}$$

(see (3.4)) and then the trigonometric functions are replaced by real variables with a polynomial constraint

$$s_k = \sin(\xi_k\, \Delta x_k)\,, \quad c_k = \cos(\xi_k\, \Delta x_k)\,, \quad s_k^2 + c_k^2 = 1\,. \tag{5.8}$$

At this stage it is important to take advantage of any special features of the problem to reduce the number of variables in the problem. For example, some problems only depend on s_k or only on c_k, while in others it is useful to eliminate the two trigonometric functions by using the tangent function

$$t_k = \tan(\xi_k \, \Delta x_k/2) \,, \quad k = 1, \cdots, d \,. \tag{5.9}$$

Note that the cases where the tangent is undefined, $\xi_k \, \Delta x_k = \pm\pi$, must be analyzed separately.

Example 5.2.1: The difference scheme is,

$$\begin{aligned} f_j^{n+1} &= \frac{1}{4}(\hat{f}_{j+1}^{n+1} + 2\hat{f}_j^{n+1} + \hat{f}_{j-1}^{n+1}) \,, \\ \hat{f}_j^{n+1} &= f_j^n - \frac{1}{2}\alpha(f_{j+1}^n - f_{j-1}^n) \,, \end{aligned}$$

which is the simple forward-time central-space with smoothing discretization of the one-way wave equation (see Strikwerda (1989) p. 47)

$$\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} = 0 \,.$$

The use of the dimensionless parameter $\alpha = a\Delta t/\Delta x$, which is called the Courant number, is important for reducing the number of variables in the problem, and is the natural variable to use for describing constraints on the time step. The variable $\hat{f}$ is used to simplify the description of the scheme and may be eliminated to produce a one-step scheme.

This scheme is one step, so the application factor of this scheme can be computed, and its square is

$$\begin{aligned} |g|^2 &= \frac{1}{4} \left[\alpha^2(-\cos^4(\xi \, \Delta x) - 2\cos^3(\xi \, \Delta x) \right. \\ &\qquad \left. +2\cos(\xi \, \Delta x) + 1) + \cos^2(\xi \, \Delta x) + 2\cos(\xi \, \Delta x) + 1 \right] \,. \end{aligned}$$

After the substitution of $c = \cos(\xi \, \Delta x)$ into the previous, the von Neumann stability condition (5.6) for this scheme can be stated as the QE problem

$$\forall c \in \mathcal{R} \left\{ -1 \leq c \leq 1 \Rightarrow \alpha^2(-c^4 - 2\,c^3 + 2\,c + 1) + c^2 + 2\,c + 1 \leq 4 \right\} \,.$$

QEPCAD solved this QE problem in 0.45 s and returns the stability condition $\alpha^2 \leq 1/2$, which agrees with the result in Strikwerda (1989) p. 48.

Example 5.2.2: In LeVeque (1994), method 3 is a six-point upwind-biased second-order accurate scheme for approximating the 2-D advection equation

$$\frac{\partial f}{\partial t} = a \frac{\partial f}{\partial x_1} + b \frac{\partial f}{\partial x_2} \,, \tag{5.10}$$

that is given by

$$\begin{aligned} f_{ij}^{n+1} = f_{ij}^n + \tfrac{1}{2} & \left[\alpha\,(f_{i+1,j}^n - f_{i-1,j}^n) + \beta\,(f_{i,j+1}^n - f_{i,j-1}^n) \right. \\ & +\alpha^2\,(f_{i+1,j}^n - 2\,f_{ij}^n + f_{i-1,j}^n) + \beta^2\,(f_{i,j+1}^n - 2\,f_{ij}^n + f_{i,j-1}^n) \\ & \left. +2\,\alpha\,\beta\,(f_{i+1,j+1}^n - f_{i+1,j}^n - f_{i,j+1}^n + f_{ij}^n) \right] \,, \end{aligned}$$

where $\alpha = a\Delta t/\Delta x_1$ and $\beta = b\Delta t/\Delta x_2$ are the Courant numbers. LeVeque (1994) shows, by numerical sampling, that the stability domain of this scheme is given by $0 \leq \alpha, \beta \leq 1$. Here we *prove* that the stability domain contains the region given by LeVeque.

The characteristic polynomial is linear in s, so it can be solved for s to give the amplification factor

$$
\begin{aligned}
g = 1 \quad &+ \quad \frac{1}{2}\left[\alpha\,(z_1 - \bar{z}_1) + \beta\,(z_2 - \bar{z}_2) + \alpha^2\,(z_1 - 2 + \bar{z}_1)\right. \\
&+ \quad \left.\beta^2\,(z_2 - 2 + \bar{z}_2) + 2\,\alpha\,\beta\,(z_1 - 1)(z_2 - 1)\right] .
\end{aligned}
$$

The square of the amplification factor can be expressed as

$$
\begin{aligned}
|g|^2 \quad = \quad & 1 + \frac{4}{(t_1^2+1)^2(t_2^2+1)^2} \\
& \left[t_1^4\,t_2^4\,\left[\alpha^4 - 4\,\alpha^3\,\beta + \alpha^2\,(6\,\beta^2 - 1) - \alpha(4\,\beta^3 + 2\,\beta) + \beta^4 - \beta^2\right]\right. \\
& +2\,t_1^4\,t_2^2\,\alpha\,\left[\alpha^3 - 2\,\alpha^2\,\beta + \alpha\,(3\,\beta^2 - 1) - 2\,\beta^2 + \beta\right] \\
& +t_1^4\,\alpha^2\,\left(\alpha^2 - 1\right) + 4\,t_1^3\,t_2^3\,\alpha\,\beta\,\left(\alpha^2 - \alpha + \beta^2 - \beta\right) \\
& +4\,t_1^3\,t_2\,\alpha^2\,\beta\,(\alpha - 1) + 2\,t_1^2\,t_2^2\,\alpha\,\beta\,[\alpha\,(3\,\beta - 2) - 2\,\beta + 1] \\
& +2\,t_1^2\,t_2^4\beta\,\left[\alpha^2\,(3\,\beta - 2) - \alpha\,(2\,\beta^2 - 1) + \beta^3 - \beta\right] \\
& \left.+\,4\,t_1\,t_2^3\,\alpha\,\beta^2\,(\beta - 1) + t_2^4\,\beta^2\,\left(\beta^2 - 1\right)\right] ,
\end{aligned}
\tag{5.11}
$$

where t_k is given by (5.9). To prove that the stability region of the scheme contains $0 \le \alpha, \beta \le 1$, it is necessary to prove that

$$
\forall\alpha \in \mathcal{R}\;\forall\beta \in \mathcal{R}\;\forall t_1 \in \mathcal{R}\;\forall t_2 \in \mathcal{R}\;\left\{(0 \le \alpha \le 1 \wedge 0 \le \beta \le 1) \Rightarrow |g|^2 \le 1\right\} . \tag{5.12}
$$

In (5.11), when $t_1 = 0$, $|g|^2 \le 1$, and consequently the substitutions $t_2 = t_1\,C_2$ and $T_1 = t_1^2$ transforms (5.12) into the equivalent form

$$
\begin{aligned}
& \forall\alpha \in \mathcal{R}\;\forall\beta \in \mathcal{R}\;\forall T_1 \in \mathcal{R}\;\forall C_2 \in \mathcal{R} \\
& \quad \left\{(0 \le \alpha \le 1 \wedge 0 \le \beta \le 1 \wedge 0 \le T_1) \Rightarrow A\,T_1^2 + B\,T_1 + C\; \le 0\right\} ,
\end{aligned}
\tag{5.13}
$$

where

$$
\begin{aligned}
A \quad &= \quad C_2^4\,(\alpha - \beta + 1)\,(\alpha - \beta - 1)\,(\alpha - \beta)^2 , \\
B \quad &= \quad 2\,C_2^4\,\beta\,\left(3\,\alpha^2\,\beta - 2\,\alpha^2 - 2\,\alpha\,\beta^2 + \alpha + \beta^3 - \beta\right) \\
& \qquad +4\,C_2^3\,\alpha\,\beta\,\left(\alpha^2 - \alpha + \beta^2 - \beta\right) \\
& \qquad +2\,C_2^2\,\alpha\,\left(\alpha^3 - 2\,\alpha^2\,\beta + 3\,\alpha\,\beta^2 - \alpha - 2\,\beta^2 + \beta\right) , \\
C \quad &= \quad C_2^4\,\beta^2\,\left(\beta^2 - 1\right) + 4\,C_2^3\,\alpha\,\beta^2\,(\beta - 1) \\
& \qquad +2\,C_2^2\,\alpha\,\beta\,(3\,\alpha\,\beta - 2\,\alpha - 2\,\beta + 1) \\
& \qquad +4\,C_2\,\alpha^2\,\beta\,(\alpha - 1) + \alpha^2\,\left(\alpha^2 - 1\right) .
\end{aligned}
$$

The discriminant d of the quadratic in (5.13) is

$$
\begin{aligned}
d \quad &= \quad 4\,\beta^2\,\alpha^2\,C_2^4\,(C_2 + 1)^2\,D , \\
D \quad &= \quad C_2^2\,R + 2\,C_2\,S + T , \\
R \quad &= \quad 8\,\alpha^2\,\beta^2 - 12\,\alpha^2\,\beta + 5\,\alpha^2 - 8\,\alpha\,\beta^3 + 8\,\alpha\,\beta^2 \\
& \qquad\qquad +2\,\alpha\,\beta - 4\,\alpha + 4\,\beta^4 - 4\,\beta^3 - 3\,\beta^2 + 4\,\beta , \\
S \quad &= \quad 4\,\alpha^3\,\beta - 2\,\alpha^3 - 4\,\alpha^2\,\beta^2 - 2\,\alpha^2\,\beta + \alpha^2 \\
& \qquad\qquad +4\,\alpha\,\beta^3 - 2\,\alpha\,\beta^2 + 2\,\alpha\,\beta - 2\,\beta^3 + \beta^2 , \\
T \quad &= \quad +4\,\alpha^4 - 8\,\alpha^3\,\beta - 4\,\alpha^3 + 8\,\alpha^2\,\beta^2 + 8\,\alpha^2\,\beta
\end{aligned}
$$

$$-3\,\alpha^2 - 12\,\alpha\,\beta^2 + 2\,\alpha\,\beta + 4\,\alpha + 5\,\beta^2 - 4\,\beta\,.$$

By calculating the extremal point of the quadratics in T_1 as in Example 5.1.1, we see that

$$\begin{aligned}&\forall\alpha\in\mathcal{R}\;\forall\beta\in\mathcal{R}\;\forall C_2\in\mathcal{R}\\&\qquad\{(0\le\alpha\le 1\wedge 0\le\beta\le 1)\Rightarrow\\&\qquad\quad A\le 0\wedge C\le 0\wedge(D\le 0\vee B\le 0)\}\end{aligned}\tag{5.14}$$

is equivalent to (5.13). Problem (5.14) can be broken into three separate parts. It is easy to see that

$$\forall\alpha\in\mathcal{R}\;\forall\beta\in\mathcal{R}\;\forall C_2\in\mathcal{R}\;\;\{(0\le\alpha\le 1\wedge 0\le\beta\le 1)\Rightarrow A\le 0\}\;,$$

and QEPCAD verified that

$$\forall\alpha\in\mathcal{R}\;\forall\beta\in\mathcal{R}\;\forall C_2\in\mathcal{R}\;\;\{(0\le\alpha\le 1\wedge 0\le\beta\le 1)\Rightarrow C\le 0\}$$

is true in 9 s. Further QEPCAD proved that

$$\forall C_2\in\mathcal{R}\;\;\{(0\le\alpha\le 1\wedge 0\le\beta\le 1)\Rightarrow B\le 0\}$$

is equivalent to

$$\begin{aligned}&\alpha\le 0\vee\alpha>1\vee\beta\le 0\vee\beta>1\vee\beta-\alpha=0\vee\\&\quad 2\,\alpha\,\beta^3-2\,\beta^3-4\,\alpha^2\,\beta^2+2\,\alpha\,\beta^2+\beta^2\\&\quad +2\,\alpha^3\,\beta+2\,\alpha^2\,\beta-5\,\alpha\,\beta+2\,\beta-2\,\alpha^3+\alpha^2+2\,\alpha-1\ge 0\end{aligned}$$

in 13 s. So statement (5.14) is proven true, except in the compliment of the set defined by the last formula, and this compliment is inside the quarter circle $\alpha^2+\beta^2\le 1/4$. So what is left to prove is that $\forall C_2\in\mathcal{R}\;\{B\le 0\vee C\le 0\}$ is true also for each α,β from this compliment which follows from

$$\begin{aligned}&\forall\alpha\in\mathcal{R}\;\forall\beta\in\mathcal{R}\;\forall C_2\in\mathcal{R}\\&\qquad\left\{(0\le\alpha\wedge 0\le\beta\wedge 4(\alpha^2+\beta^2)<1)\Rightarrow(B\le 0\vee D\le 0)\right\}\;.\end{aligned}$$

QEPCAD proved this formula is true, but this took over 9 hours, using 16 MB of memory, and almost reached the limits of the QEPCAD package. QEPCAD was unable to prove that formula (5.14) is true or even that the same formula with $A\le 0\wedge C\le 0$ skipped is true. Thus it was necessary to limit the searched area to $\alpha^2+\beta^2<1/4$ to be able to proceeded. This is a new result because LeVeque derived this stability domain using numerical sampling, but did not prove that the estimated domain was, in fact, in the stability region.

6. Initial-Boundary Value Problems for Partial Differential Equations

The problems in this section are similar to the problems in Section 5.1 on initial-value problems for PDEs, except one of the spatial variables is restricted to a half space and then a boundary condition is given on the face of the half space. It is assumed that the initial-value problem for the system of PDEs is stable, and then the stability of the boundary conditions is analyzed. The analysis of the boundary conditions does not follow the pattern of the solution of the previous problems, so more details are given.

6.1. Continuous Equations

In the case of initial-boundary value problems for PDEs, as before, $t > 0$ and $\vec{x} \in \mathcal{R}^d$, but now there is an additional spatial variable $y \geq 0$, that is, the the PDEs are defined on a half space in $\mathcal{R}^{d+1}$:

$$\mathbf{S}(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\, \vec{f}(t, y, \vec{x}) = 0\,, \tag{6.1}$$

where $\vec{f} = \vec{f}(t, y, \vec{x})$ is a N-component vector of complex-valued functions of time $t \geq 0, y \geq 0$ and $\vec{x} \in \mathcal{R}^d$. The boundary conditions (BCs) are given at $y = 0$, and there do not have to be the same number as there are equations in the system. Thus let $M \geq 0$ be an integer and then let $\mathbf{B}$ be a $M \times N$ matrix of complex polynomials in the variables ∂_t, ∂_y, and $\partial_{\vec{x}}$. The boundary conditions are then given by

$$\mathbf{B}(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\, \vec{f}(t, y, \vec{x})\Big|_{y=0} = 0\,. \tag{6.2}$$

The trial solution for the initial-boundary value problem has the form

$$\vec{f}(t, y, \vec{x}) = e^{\lambda\, t + \eta\, y + \imath\, \vec{\xi}\cdot\vec{x}}\, \vec{v}(\lambda, \eta, \imath\, \vec{\xi})\,, \tag{6.3}$$

where $\lambda\,,\, \eta \in \mathcal{C}$ and $\vec{\xi} \in \mathcal{R}^d$. For $\vec{f}$ to be a non-trivial solution of the PDEs, it must be the case that

$$C(\lambda, \eta, \vec{\xi}, \vec{\alpha}) = \det \mathbf{S}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha}) = 0\,, \quad \mathbf{S}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha})\, \vec{v}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha}) = 0\,, \tag{6.4}$$

where the null vector, that is, the eigenvector for eigenvalue zero, $\vec{v}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha})$, is not zero.

For initial-boundary value problems, we only check to see if they have unstable solutions, that is, do they posses nontrivial solutions that are bounded in space and unbounded in time, which means that

$$\Re\, \lambda > 0\,, \quad \Re\, \eta < 0\,, \quad \vec{\xi} \in \mathcal{R}^d\,. \tag{6.5}$$

To have stability, the boundary conditions should eliminate such solutions. Note that the roots with both $\Re\lambda > 0$ and $\Re\eta > 0$ give solutions that are growing and time, but also grow in space, so they are not used to test stability.

The number of unstable solutions is computed by counting the number of roots $\eta_k = \eta_k(\lambda, \vec{\xi}, \vec{\alpha})$ of the characteristic polynomial for which the conditions (6.5) holds. The stability of the initial value problem implies that the number K of roots satisfying (6.5) is constant, so this number may be computed by choosing convenient values for η and $\vec{\xi}$ and then computing the roots of the characteristic polynomial. The null vectors $\vec{v}$ from (6.4) corresponding to these roots are denoted by $\vec{v}_k(\lambda, \eta_k, \vec{\xi}, \vec{\alpha})$. A linear combination of the unstable solutions,

$$\vec{f}(t, x, \vec{y}) = \sum_{k=1}^{K} c_k\, e^{\lambda\, t + \eta_k(\lambda, \vec{\xi}, \vec{\alpha})\, y + \imath\, \vec{\xi}\, \vec{x}}\, \vec{v}_k(\lambda,\, \eta_k,\, \vec{\xi},\, \vec{\alpha})\,, \tag{6.6}$$

are chosen as trial solutions for the boundary conditions (6.2). We assume that all the roots $\eta_k, k = 1, \cdots, K$ are simple, otherwise the form of (6.6) is not correct.

If the boundary conditions (6.2) allow a non-trivial unstable solution (6.6) then the BCs are unstable, and consequently the initial-boundary value problem is unstable. To see if such solutions exit, substitute (6.6) into the BCs (6.2) and divide out the exponentials to get

$$\sum_{k=1}^{K} c_k\, \mathbf{B}(\lambda,\, \eta_k(\lambda,\, \vec{\xi},\, \vec{\alpha}),\, \imath\vec{\xi})\, \vec{v}_k(\lambda,\, \eta_k, \vec{\xi},\, \vec{\alpha}) = 0\,, \tag{6.7}$$

which can be written as

$$\mathbf{T}\,\vec{c} = 0 \tag{6.8}$$

where

$$\mathbf{T}(\lambda,\, \vec{\eta},\, \vec{\xi},\, \vec{\alpha}) = \left(\mathbf{B}\left(\lambda,\, \eta_1,\, \imath\vec{\xi},\, \vec{\alpha}\right) \vec{v}_1(\lambda,\, \eta_1,\, \vec{\xi},\, \vec{\alpha}), \cdots, \mathbf{B}\left(\lambda,\, \eta_K,\, \imath\vec{\xi},\, \vec{\alpha}\right) \vec{v}_K(\lambda,\, \eta_K,\, \vec{\xi},\, \vec{\alpha})\right)$$

and $\vec{c} = (c_1,\, \cdots,\, c_K)$, $\vec{\eta} = (\eta_1,\, \cdots,\, \eta_K)$, and $\mathbf{T}$ is a M by K matrix.

The system $\mathbf{T}\,\vec{c} = 0$ always has a non-trivial solution $\vec{c} \neq 0$ if $M < K$. If $M > K$ there are too many boundary conditions, so we assume that the number of boundary conditions is the same as the number of unstable roots, that is, $M = K$. Now there are unstable solutions if and only if $\mathbf{T}$ is not of full rank: $\mathrm{rank}(\mathbf{T}) < K$.

Thus the existence of non-trivial unstable solutions of the form (6.6) with (6.5) is implied by the QE problem

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\ \exists\vec{\eta} \in \mathcal{C}^K\ \exists\vec{\xi} \in \mathcal{R}^d\ \exists\vec{v}_1 \in \mathcal{R}^N\ \cdots\exists\vec{v}_K \in \mathcal{R}^N \\ &\quad \{\Re\lambda > 0 \wedge \Re\vec{\eta} < 0 \wedge \mathrm{Distinct}(\vec{\eta})\wedge \\ &\qquad \bigwedge_{k=1}^{K} \{C(\lambda,\, \eta_k,\, \vec{\xi},\, \vec{\alpha}) = 0 \wedge \mathbf{S}(\lambda,\, \eta_k,\, \imath\vec{\xi},\, \vec{\alpha})\, \vec{v}_k = 0 \wedge \vec{v}_k \neq 0\} \wedge \\ &\qquad \det(\mathbf{T}(\lambda,\, \vec{\eta},\, \vec{\xi},\, \vec{\alpha})) = 0\Big\}\,, \end{aligned} \tag{6.9}$$

where

$$\mathrm{Distinct}(\vec{\eta}) = \bigwedge_{i,j=1}^{K} i \neq j \Rightarrow \eta_i \neq \eta_j\,. \tag{6.10}$$

In summary, the QE problem (6.9) defines sufficient conditions for the instability of the boundary conditions (6.2), that is, if (6.9) is true then the BCs (6.2) are unstable. Condition (6.9) defines a set in the space of parameters $\vec{\alpha}$ inside of which the BCs are unstable.

If the system of PDEs consists of only one scalar PDE, then $\mathbf{S} = (C)$, $\mathbf{B}$ is K by 1 matrix, and statement (6.9) becomes

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\ \exists\vec{\eta} \in \mathcal{C}^K\ \exists\vec{\xi} \in \mathcal{R}^d \\ &\quad \Big\{\Re\lambda > 0 \wedge \Re\vec{\eta} < 0 \wedge \mathrm{Distinct}(\vec{\eta}) \wedge \bigwedge_{k=1}^{K} C(\lambda,\, \eta_k,\, \vec{\xi},\, \vec{\alpha}) = 0 \\ &\qquad \wedge \det(\mathbf{B}(\lambda,\, \eta_1,\, \vec{\xi},\, \vec{\alpha}),\, \cdots,\, \mathbf{B}(\lambda,\, \eta_K,\, \vec{\xi},\, \vec{\alpha})) = 0\Big\}\,. \end{aligned} \tag{6.11}$$

Example 6.1.1: The heat or diffusion equation with Robin BCs is

$$\frac{\partial f}{\partial t} = \alpha\, \frac{\partial^2 f}{\partial x^2} + \beta\, \frac{\partial^2 f}{\partial^2 y}\,, \quad \left(a\, \frac{\partial f}{\partial y} + b\, f\right)_{y=0} = 0\,, \tag{6.12}$$

where $\alpha, \beta, a, b \in \mathcal{R}$, $a\, b \neq 0$. The initial value problem is stable if and only if $\alpha \geq 0$ and

$\beta \geq 0$. For this problem

$$C(\lambda,\, \eta) = \lambda + \alpha\, \xi^2 - \beta\, \eta^2\,, \quad B(\lambda,\, \eta) = a\, \eta + b\,. \tag{6.13}$$

To count the number of boundary conditions, note that the roots in η of $C(\lambda, \eta) = 0$ are

$$\eta = \pm\sqrt{\frac{\lambda + \alpha\, \xi^2}{\beta}}\,, \tag{6.14}$$

so $K = 1$.

The existence of exponential unstable solutions for the IBVP is implied by

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\ \exists\eta \in \mathcal{C}\ \exists\xi \in \mathcal{R} \\ &\qquad \left\{\Re\lambda > 0 \wedge \Re\eta < 0 \wedge \lambda + \alpha\, \xi^2 - \beta\, \eta^2 = 0 \wedge a\, \eta + b = 0\right\}\,. \end{aligned} \tag{6.15}$$

As is well known and easy to check, the IBVP has exponential unstable solutions for $a\, b > 0$. In 28 s, QEPCAD calculated the quantifier free formula equivalent to (6.15):

$$\begin{gathered} ((a < 0 \wedge b < 0 \wedge \beta \neq 0) \vee (a = 0 \wedge b = 0 \wedge \beta \neq 0) \vee (a > 0 \wedge b > 0 \wedge \beta \neq 0)) \\ \wedge\alpha \geq 0 \wedge \beta \geq 0\,, \end{gathered}$$

which is equivalent to $\alpha \geq 0 \wedge \beta > 0 \wedge (a\, b > 0 \vee a = b = 0)$, which gives the region of instability.

Example 6.1.2: This example is a simple IBVP for a hyperbolic system that is Example 11.4.1 on p. 260 in Strikwerda (1989) and Example 8.4.1 on p.307 in Kreiss and Lorenz (1989):

$$u_t + u_y - v_x = 0\,, \quad v_t - v_y - u_x = 0\,, \quad (u + a\, v)_{y=0} = 0\,, \tag{6.16}$$

where $a \in \mathcal{C}$. For this example the matrix $\mathbf{S}$ is given by

$$\mathbf{S}(\lambda,\, \eta,\, \imath\, \xi) = \begin{pmatrix} \lambda + \eta & -\imath\, \xi \\ -\imath\, \xi & \lambda - \eta \end{pmatrix} \tag{6.17}$$

and then

$$C(\lambda,\, \eta,\, \xi) = \lambda^2 - \eta^2 + \xi^2\,. \tag{6.18}$$

The matrix $\mathbf{B}$ for the boundary conditions is the 1×2 matrix

$$\mathbf{B}(\lambda,\, \eta,\, \imath\, \xi) = (1,\, a)\ . \tag{6.19}$$

To count the number of boundary conditions, solve $C = 0$ for η:

$$\eta = \pm\sqrt{\lambda^2 + \xi^2}\,. \tag{6.20}$$

so there is one root with $\Re\, \eta > 0$ and one root with $\Re\, \eta < 0$, and consequently $K = 1$. For the solutions of (6.20), the matrix $\mathbf{S}$ has the null vectors

$$\vec{v}_1 = \begin{pmatrix} \lambda - \eta \\ \imath\, \xi \end{pmatrix}\,, \qquad \vec{v}_2 = \begin{pmatrix} \imath\, \xi \\ \lambda + \eta \end{pmatrix}\,. \tag{6.21}$$

In this case the matrix $\mathbf{T}$ is a scalar (T). If we use the first null vector $\vec{v}_1$, then $T = \mathbf{B}\vec{v}_1 = \lambda - \eta + \imath\, a\, \xi$. Instability requires $T = 0$, which cannot be achieved for the root with $\Re\eta < 0$ and $\Re\lambda > 0$. So we have to use the null vector $\vec{v}_2$, in which case $T = \imath\, \xi + a\, (\lambda + \eta)$ and the existence of exponential unstable solutions is implied by

$$\exists\lambda \in \mathcal{C}\ \exists\eta \in \mathcal{C}\ \exists\xi \in \mathcal{R}$$

$$\{\Re\lambda > 0 \wedge \Re\eta < 0 \wedge \lambda^2 - \eta^2 + \xi^2 = 0 \wedge (\xi \neq 0 \vee \lambda + \eta \neq 0) \wedge \\ \imath\xi + a\,(\lambda + \eta) = 0\}\ . \tag{6.22}$$

The equations in the previous condition can be solved for λ, η, and ξ and the result shows that there are no exponential unstable solutions for any value of $a \in \mathcal{R}$. For $a \in \mathcal{R}$, QEPCAD solved this QE problem, using equational constraints (Collins (1994)) , in less then a second, giving the result false, i.e. the BC is stable. Using equational constraints is important because, when the constraints are used, QEPCAD finishes the calculation almost immediately using only 4 MB of memory, while without the constraints, QEPCAD fails after more than 5 hours of CPU time using 64 MB of memory. For complex $a \in \mathcal{C}$, one can show by hand calculations that the solution of the above QE problem is $|a| > 1 \wedge \Im a \neq 0$. QEPCAD was unable to solve this case, even after the variable ξ was eliminated.

Example 6.1.3: This example is presented to give the reader some idea of how difficult problems are that occur in realistic applications. Hagstrom and Lorenz (1993), derive and analyze some boundary conditions for incompressible, low Mach-number flows. We do not reproduce the differential equations or boundary conditions here, but only give the ingredients of the QE problem. The matrix for the system of PDEs is

$$\mathbf{S}(\lambda,\,\eta,\,\imath\vec{\xi}) = \begin{pmatrix} X & \imath\xi & \eta \\ \imath\xi & X - \epsilon\,\eta^2 + \frac{4}{3}\,\epsilon\,\xi^2 & \frac{-\imath}{3}\,\epsilon\,\xi\,\eta \\ \eta & \frac{-\imath}{3}\,\epsilon\,\xi\,\eta & X - \frac{4}{3}\,\epsilon\,\eta^2 + \epsilon\,\xi^2 \end{pmatrix}, \tag{6.23}$$

where $X = \lambda + \eta\,V + \imath\,U\,\xi$, $\epsilon \geq 0$ and U and V are real parameters. The matrix for one set of boundary conditions is

$$\mathbf{B}(\lambda,\,\eta,\,\imath\vec{\xi}) = \begin{pmatrix} \imath\xi & X & 0 \\ -\lambda - |\xi| & -\imath\,V\,\xi & \lambda + \imath\,U\,\xi \end{pmatrix}. \tag{6.24}$$

A sufficient condition for instability (6.9) is given by the QE problem

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\ \exists\eta_1 \in \mathcal{C}\ \exists\eta_2 \in \mathcal{C}\ \exists\vec{\xi} \in \mathcal{R}\ \exists\vec{v}_1 \in \mathcal{R}^3\ \exists\vec{v}_2 \in \mathcal{R}^3 \\ &\quad\{\Re\lambda > 0 \wedge \Re\eta_1 < 0 \wedge \Re\eta_2 < 0 \wedge \eta_1 \neq \eta_2 \wedge \\ &\qquad \det\mathbf{S}(\eta_1) = 0 \wedge \mathbf{S}(\eta_1)\vec{v}_1 = 0 \wedge \vec{v}_1 \neq 0 \wedge \\ &\qquad \det\mathbf{S}(\eta_2) = 0 \wedge \mathbf{S}(\eta_2)\vec{v}_2 = 0 \wedge \vec{v}_2 \neq 0 \wedge \\ &\qquad \det(\mathbf{B}(\eta_1)\vec{v}_1, \mathbf{B}(\eta_2)\vec{v}_2) = 0\}\ , \end{aligned}$$

6.2. Semi-discrete Equations

In the semi-discrete approach the PDEs and BCs are discretized in the spatial coordinates, while time is kept continuous, so the discrete functions are given by

$$\vec{f}_{m,\vec{j}}(t) \approx \vec{f}(t, m\Delta y, \vec{j} * \Delta\vec{x})\,, \tag{6.25}$$

where $t \geq 0$, $m \geq 0$ and $\vec{j}$ an arbitrary multi-index. A general linear semi-discrete finite-difference scheme approximating (6.1) and (6.2) can be written in the form

$$\mathbf{S}(\partial_t,\,\delta_y,\,\delta_{\vec{x}},\,\vec{\alpha})\vec{f}^n_{m,\vec{j}} = 0\,, \quad \mathbf{B}(\partial_t,\,\delta_y,\,\delta_{\vec{x}},\,\vec{\alpha})\vec{f}^n_{0,\vec{j}} = 0 \tag{6.26}$$

where $\mathbf{S}$ is a $N \times N$ matrix, and $\mathbf{B}$ is an $M \times N$ matrix of polynomials in ∂_t, δ_y, and $\delta_{\vec{x}}$. Note that $\mathbf{S}$ and $\mathbf{B}$ are not the same as in (6.1) and (6.2). It is assumed that $\text{rank}(\mathbf{B}) = M$ when $\mathbf{B}$ is considered as a matrix over the ring of polynomials.

The trial solutions for the semi-discrete PDEs initial-boundary value problem has the form

$$\vec{f}_{m,\vec{j}}(t) = e^{\lambda\, t}\, w^m\, \vec{z}^{\vec{j}}\, \vec{v}\,, \quad |z_k| = 1\,, \quad k = 1, \cdots, d\,, \tag{6.27}$$

where $t \geq 0$, $m \geq 0$, $\vec{j}$ is an arbitrary multi-index, $\lambda \in \mathcal{C}$, $w \in \mathcal{C}$, and $\vec{z} \in \mathcal{C}^d$. The trial solution $\vec{f}_{m,\vec{j}}(t)$ is a non-trivial solution of the interior scheme (6.26) if and only if

$$C(\lambda,\, w,\, \vec{z},\, \vec{\alpha}) = \det \mathbf{S}(\lambda,\, w,\, \vec{z},\, \vec{\alpha}) = 0\,, \quad \mathbf{S}(\lambda,\, w,\, \vec{z},\, \vec{\alpha})\vec{v}(\lambda,\, w,\, \vec{z},\, \vec{\alpha}) = 0\,, \tag{6.28}$$

where the null vector $\vec{v}(\lambda, w, \vec{z}, \vec{\alpha}) \neq 0$.

For the semi-discrete initial-boundary value problems, we only check to see if they have unstable solutions, that is, do they posses nontrivial solutions that are bounded in space and unbounded in time, which means that

$$\Re\lambda > 0\,, \quad |w| < 1\,, \quad |z_k| = 1\,, \quad k = 1, \cdots, d\,. \tag{6.29}$$

To have stability, the boundary conditions should eliminate such solutions. The analysis of the semi-discrete problems is very similar to analysis in the previous section, only the left half-plane in η is replaced by interior of the unit circle in w.

Now the number of boundary conditions is given by solving the characteristic polynomial for w. The assumed stability of the discrete initial value problem implies that the number of such roots is a constant K. The roots are written as $w_k = w_k(\lambda, \vec{z}, \vec{\alpha})$, $k = 1, \cdots, K$ and the null vectors $\vec{v}$ corresponding to these roots are denoted by $\vec{v}_k(\lambda, w_k, \vec{z}, \vec{\alpha})$. A linear combination of the unstable solutions,

$$\vec{f}_{m,\vec{j}}(t) = \sum_{k=1}^{K} c_k\, e^{\lambda\, t}\, w_k^m(\lambda,\, \vec{z},\, \vec{\alpha})\, \vec{z}^{\vec{j}}\, \vec{v}_k(s,\, w_k,\, \vec{z},\, \vec{\alpha})\,, \tag{6.30}$$

are chosen as trial solutions for the boundary conditions. Again, we assume that all the roots $w_k, k = 1, \cdots, K$ are simple, otherwise the form of the trial solutions (6.30) is not correct. If the boundary conditions (6.26) allow a non-trivial (i.e. non-zero) unstable solution (6.30) then the BCs are unstable and consequently the initial-boundary value problem is unstable.

Substituting the trial solution (6.30) into the BCs $\mathbf{B}$ and dividing by the trial solution gives

$$\mathbf{T}\, \vec{c} = 0 \tag{6.31}$$

where

$$\mathbf{T}(\lambda,\, \vec{w},\, \vec{z},\, \vec{\alpha}) = (\mathbf{B}\,(\lambda,\, w_1,\, \vec{z},\, \vec{\alpha})\, \vec{v}_1(\lambda,\, w_1,\, \vec{z},\, \vec{\alpha}), \cdots, \mathbf{B}\,(\lambda,\, w_K,\, \vec{z},\, \vec{\alpha})\, \vec{v}_K(\lambda,\, w_K,\, \vec{z},\, \vec{\alpha})\,)$$

and $\vec{c} = (c_1, \cdots, c_K)$, $\vec{w} = (w_1, \cdots, w_K)$, and $\mathbf{T}$ is a M by K matrix.

The conditions for instability are essentially the same as in the previous section. The existence of non-trivial unstable solutions of the form (6.30) with (6.29) is implied by the QE problem

$$\begin{aligned}
&\exists\lambda \in \mathcal{C}\; \exists\vec{w} \in \mathcal{C}^K\; \exists\vec{z} \in \mathcal{R}^d\; \exists\vec{v}_1 \in \mathcal{R}^N\; \cdots \exists\vec{v}_K \in \mathcal{R}^N \\
&\quad \{\Re\lambda > 0 \wedge |\vec{w}| < 1 \wedge \mathrm{Distinct}(\vec{w}) \wedge |\vec{z}| = 1 \wedge \\
&\quad \bigwedge_{k=1}^{K} \{C(\lambda,\, w_k,\, \vec{z},\, \vec{\alpha}) = 0 \wedge \mathbf{S}(\lambda,\, w_k,\, \vec{z},\, \vec{\alpha})\, \vec{v}_k = 0 \wedge \vec{v}_k \neq 0\} \wedge \\
&\quad \det(\mathbf{T}(\lambda,\, \vec{w},\, \vec{z},\, \vec{\alpha})) = 0\}\,.
\end{aligned} \tag{6.32}$$

In summary, the QE problem (6.32) defines a sufficient condition for the instability of the boundary conditions (6.26), and thus defines a region in the space of parameters $\vec{\alpha}$ inside which the BCs are unstable.

If the system of PDEs is in 1-D (i.e. there is only one space variable y), and if it consists of only one scalar PDE, then $\mathbf{S} = (C)$, $\mathbf{B}$ is K by 1 matrix, and the statement (6.32) becomes

$$\begin{aligned}&\exists\lambda \in \mathcal{C}\ \exists\vec{w} \in \mathcal{C}^K \\ &\qquad\Big\{\Re\lambda > 0 \wedge |\vec{w}| < 1 \wedge \mathrm{Distinct}(\vec{w}) \wedge \bigwedge_{k=1}^{K} C(\lambda,\, w_k,\, \vec{\alpha}) = 0 \\ &\qquad\qquad \wedge \det(\mathbf{B}(\lambda,\, w_1,\, \vec{\alpha}),\, \cdots,\, \mathbf{B}(\lambda,\, w_K,\, \vec{\alpha})) = 0\Big\}\ .\end{aligned} \tag{6.33}$$

Example 6.2: Carpenter, Gottlieb and Abarbanel (1993) study boundary conditions for compact higher-order finite-difference schemes that approximate the scalar one-dimensional transport equation using the semi-discrete approach. Their fourth-order semi-discretization for the inflow problem is defined by

$$f'_{m+1} + 4\, f'_m + f'_{m-1} + \frac{3\,(f_{m+1} - f_{m-1})}{\Delta y} = 0$$

at the interior points (here f' is the time derivative of f). On p. 281 they analyze the combined first- and second-order boundary condition

$$\left[2\,(1+2\,\beta) - \frac{1}{2}\right] f'_1 + \frac{1+2\,\beta}{2}\, f'_2 + \frac{1}{2\,\Delta y}\,[-2\,(1+\beta)\, f_1 + 3\,(1+2\beta)\, f_2] = 0\,,$$

The characteristic polynomial of the interior scheme is

$$C(\Lambda,\, w) = (1 + 4\, w + w^2)\Lambda + 3\,(w^2 - 1),$$

where $\Lambda = \lambda\Delta y$, and the characteristic polynomial for the boundary condition is

$$B(\Lambda,\, w) = \left[2\,(1+2\,\beta) - \frac{1}{2} + \frac{1+2\,\beta}{2}\, w\right]\Lambda - 1 - \beta + \frac{3}{2}\,(1+2\,\beta)\, w\,.$$

The existence of the unstable solutions is implied by the QE problem

$$\exists\Lambda \in \mathcal{C}\ \exists w \in \mathcal{C}\ \{\Re\Lambda > 0 \wedge |w| < 1 \wedge C(\Lambda,\, w) = 0 \wedge B(\Lambda,\, w) = 0\}\,. \tag{6.34}$$

If the equation $C(\Lambda, w) = 0$ is solved for Λ and the solution is substituted into the equation $B(\Lambda, w) = 0$, the $|w| < 1$ implies that $\Im w = 0$. So, with $r = \Re\, w$, the QE problem is equivalent to the simpler one

$$\exists r\ \{-1 < r \wedge r < 1 \wedge r^2 + 4\, r + 1 > 0 \wedge r^2\,(2\,\beta - 1) + 2\, r\,(-2\,\beta + 1) - 22\,\beta - 7 = 0\}\,,$$

which QEPCAD solves in less than 1 s yielding

$$4\,\beta + 1 < 0 \wedge 2\,\beta^2 - 2\,\beta - 1 < 0\,.$$

So if this condition is true, then the boundary condition is unstable. Solving the quadratic gives the approximate interval of instability as $-0.366 < \beta < -0.25$ which is the same as the result of Carpenter, Gottlieb and Abarbanel (1993).

7. Conclusion

We have shown that the techniques presented in this paper can be used to analytically solve simple stability problems in a few seconds, and most textbook examples in from a few minutes to a few hours. Even some research problems can be handled. On the other hand, there are some relatively modest problems beyond the reach of our techniques, that is, the problem cannot be solved directly by QEPCAD or the intermediate results produced by the Routh-Hurwitz type algorithms are too complicated to be solved by QEPCAD. The main difficulty is typically that the number of real variables in the problem is too large for QEPCAD to handle. Thus many problems of practical importance are beyond our capabilities, even if we use the one of the largest computers to do our calculations.

However, one should keep in mind that this is a new research area, both in terms of applications, and in terms of the development of quantifier elimination algorithms. In fact many groups, as can be seen by looking at our references, are now working on better quantifier-elimination algorithms. Some are working on special cases such as those containing low degree polynomials and other are working on completely solving some special problems so that the solutions can be used to speed up other algorithms. We are especially interested the notion of approximate quantifier-elimination where only the most important cases in a problem are considered. This notion seem particularly relevant to certain stability problems, and promises drastic speed-ups of the algorithms. In any case, we expect substantial improvements in the quantifier-elimination algorithms and significant speed-up of computer, which will allow us to solve even more important problems.

In our current approach, we typically completely solve a given quantifier-elimination problem. For even modest problems, the answer can be so complex as to be useless. On a little reflection, this seems natural. The stability regions are complicated and consequently their description using polynomial inequalities must be complicated. From an applications point of view, a description of the complete stability region is often not necessary. What is usually needed is some good approximation of the stability region or even a good estimate of the largest ball or box contained in the stability region. There is much to be done on understanding how to formulate quantifier-elimination problems so that the useful answer can be obtained for complicated stability problems.

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A. Notation

Our notation is summarized in Table 2. As usual, several different types of notion are used for different stability problems. We use a consistent notation throughout all of the sections, so the material on boundary-value problems, being the most complex section, determined the notation.

Table 2. Notation for Differential and Difference Equations

quantity	meaning
d	number of spatial variables, $d \in \mathcal{Z}$, $d \geq 0$
t	time, $t \in \mathcal{R}$, $t \geq 0$
$\vec{x}$	spatial variables without boundary conditions, $\vec{x} \in \mathcal{R}^d$
y	spatial variable with a boundary condition, $y \in \mathcal{R}$, $y \geq 0$
λ	Laplace transform variable in t, $\lambda \in \mathcal{C}$
$\vec{\xi}$	Fourier transform variables for $\vec{x}$, $\vec{\xi} \in \mathcal{R}^d$
η	Laplace transform variable for y, $\eta \in \mathcal{C}$
Δt	time step
$\Delta\vec{x}$	space grid steps in variables $\vec{x}$
Δy	space grid step in variable y
n	index for discrete time variable t, $n \in \mathcal{Z}$, $n \geq 0$
$\vec{j}$	indices for discrete spatial variables $\vec{x}$, $\vec{j} \in \mathcal{Z}^d$
m	index for discrete spatial variable y, $m \in \mathcal{Z}$, $m \geq 0$
s	discrete Laplace variable for t, $s \in \mathcal{C}$
$\vec{z}$	discrete Fourier variable for $\vec{x}$, $\vec{z} \in \mathcal{C}$, $\lvert\vec{z}\rvert = 1$
w	discrete Laplace variable for y, $w \in \mathcal{C}$
N, M	number of DEs and BCs, $N, M \in \mathcal{Z}$
$\vec{f}$	N-component vector of complex-valued functions
$\mathbf{S}$, $\mathbf{B}$	matrices describing the DEs and BCs
$\vec{\alpha}$	real parameters of the stability problem
$\mathbf{I}$	unit matrix
μ, $\vec{v}$	eigenvalues and eigenvectors
C	characteristic polynomial
$\mathcal{R}$, $\mathcal{C}$, $\mathcal{Z}$	real numbers, complex numbers, integers
z, $\bar{z}$	complex variable, complex conjugate
$\imath$	complex unit, $\imath^2 = -1$
$\Re$, $\Im$	real part, imaginary part
e	Euler number

The standard multi-index notation is used

$$\begin{aligned}
\vec{j} &= (j_1, \cdots j_d) \\
\vec{x} &= (x_1, \cdots x_d) \\
\vec{j} * \vec{x} &= (j_1\, x_1, \cdots, j_d\, x_d) \\
\vec{x} \cdot \vec{y} &= x_1\, y_1 + \cdots + x_d\, y_d \\
\vec{x}^{\vec{j}} &= {x_1}^{j_1} \cdots {x_d}^{j_d} \\
\partial_{\vec{x}} &= (\partial_1, \cdots, \partial_d)
\end{aligned}$$

The notation for partial derivatives is:

$$\partial_k = \frac{\partial}{\partial_{x_k}}\,, \quad \partial_t = \frac{\partial}{\partial t}\,, \quad \partial_y = \frac{\partial}{\partial y}\,.$$

If a continuum problem is discretized, then is is assumed that $\vec{f}^n_{m,\vec{j}}$ approximates $\vec{f}(n\Delta t,\, m\,\Delta y,\, \vec{j} * \Delta\vec{x})$ on an uniform grid, and that $n \geq 0$, $m \geq 0$, while $\vec{j}$ can be an arbitrary integer multi-index. Then the shift operator δ is defined by

$$\begin{aligned}
\delta_t \vec{f}^n_{m,\vec{j}} &= \vec{f}^{n+1}_{m,\vec{j}}\,, \\
\delta_y \vec{f}^n_{m,\vec{j}} &= \vec{f}^n_{m+1,\vec{j}}\,, \\
\delta_{x_k} \vec{f}^n_{m,\vec{j}} &= \vec{f}^n_{m,\vec{j}+\vec{1}_k}\,, \qquad k = 1,\cdots,d\,, \\
\delta_{\vec{x}} &= (\delta_{x_1},\cdots,\delta_{x_d})
\end{aligned} \tag{A.1}$$

where

$$\vec{1}_1 = (1,0,\cdots,0),\ \vec{1}_2 = (0,1,\cdots,0),\ \cdots,\ \vec{1}_d = (0,0,\cdots,1)\,.$$