Synchronized flux corrected remapping for ALE methods

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A R T I C L E    I N F O

Article history:
Received 30 April 2010
Received in revised form 9 November 2010
Accepted 16 November 2010
Available online 21 November 2010

Keywords:
ALE
FCT
Remapping

A B S T R A C T

A new optimization-based synchronized flux corrected conservative interpolation (remapping) of mass, momentum and energy for arbitrary Lagrangian Eulerian method is developed. Fluxes of conserved variables (mass, momentum and total energy) are limited in a synchronous FCT-like way to preserve local bounds in density, velocity and specific internal energy.

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1. Introduction

The arbitrary Lagrangian Eulerian (ALE) method [1] is widely used in hydrodynamical fluid flow modeling. It consists of several Lagrangian steps followed by rezoning and remapping steps. Rezoning step smooths the Lagrangian mesh to prevent its distortion and remapping performs conservative interpolation of the conserved quantities from the old Lagrangian mesh to the new rezoned one. To minimize interpolation errors one needs to use the high-order remapping, which however tends to produce overshoots or undershoots (i.e. creates new local extrema) in the vicinity of discontinuities. Creation of the new local extrema can be reduced by using slope limiters during reconstruction phase of the remapping [2], however this approach in general cannot avoid new local extrema. To remove the new local extrema one typically employs some a posterior procedure such as repair [3] which distributes the excessive conservative quantity into the neighboring cells.

To avoid this a-posteriori correction we propose a new remapping method, based on the flux corrected transport (FCT) ideas [4–6], which does not produce new local extrema. The new flux corrected remap (FCR) is local by definition (one interface at a time), simple and fast. Its extension to general multidimensional meshes is easy. The FCT type methods have been applied to system of Euler equations by either FCT limiting of characteristic variables [7,8] or by using the same limiter (being the minimum of density and energy or density and pressure limiters) for all equations [9,10], which is sometimes called synchronized. By synchronized we however understand something else. We have three independent FCR limiters for conserved mass, momentum and energy and we choose these limiters in such a way that the remapped solution satisfies bounds in density, velocity and internal energy and our FCR solution stays close to the high-order solution. Schär and Smolarkiewicz [11] apply the synchronized FCT method to the system of transport equations and we compare our FCR with remapping based on their method. In the FCR method we guarantee the bounds in density, velocity and specific internal energy (internal energy per unit mass) by constraining the fluxes of mass, momentum and total energy. We employ independent flux limiting correction factors for mass, momentum and total energy. From the bounds we derive the constraints on the correction factors. The constrained optimization chooses the correction factors satisfying the constraints which are in some sense optimal, i.e. producing fluxes close to the high-order fluxes. Compared to the sequential FCR [12], we apply the bounds simultaneously to treat all quantities in the system at once, thus we call our method synchronized FCR (SFCR). The SFCR method has been developed in [13] for mass and momentum with bounds in density and velocity. Here we extend it to include also remap of total energy with bounds in internal energy.

2. Problem statement

The underlying functions describing the fluid at position \( r \) are density \( \rho(r) \), velocity \( \mathbf{u}(r) \), specific internal energy \( e(r) \) and total energy \( e(r) = \rho(r + u^2)/2 \). The rezoning phase of the ALE method constructs from the old Lagrangian mesh \( \Omega \) (consisting of cells \( \Omega_i \)) the new rezoned mesh \( \bar{\Omega} \) (consisting of cells \( \bar{\Omega}_i \)). The volume \( V_0 \), mass \( m_0 \), momentum \( \mathbf{p} \) and total energy \( E_t \) of the old mesh cells

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\( \Omega \) are given by \( V_i = \int_{\Omega_i} dV \), \( m_i = \int_{\Omega_i} \rho dV \), \( \mu_i = \int_{\Omega_i} \rho \mathbf{u} dV \), 
\( E_i = \int_{\Omega_i} \rho dV \), and define the mean values of density, velocity and 
internal energy in the old mesh cells \( \Omega_i \) as \( \bar{\rho}_i = \bar{m}_i/V_i, \bar{\mu}_i = \bar{\rho}_i \bar{u}_i, \)
\( \bar{E}_i = E_i/m_i \), and the question is how to choose correction factors \( 0 \leq C_{\min} \leq 1 \) for 
which the remapped values satisfy the bounds. Such correction factor always exists 
as the low-order scheme (obtained for \( C_{\min} = 0 \)) preserves the local bounds.
With anti-diffusive fluxes \( \mathbf{dF}^{\text{em}} = F^{\text{em}H} - F^{\text{em}L} \) the new mass obtained 
tained by FCR can be written as
\[
\hat{m}_i^{\text{FCR}} = m_i + \sum_k C_{\min}^{\text{FCR}} = \hat{m}_i^{\text{max}} + \sum_k C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k.
\]
The density constraints \( \rho_i^{\text{min}} \leq \hat{\rho}_i \leq \rho_i^{\text{max}} \) are transformed into mass constraints 
\( m_i^{\text{max}} \leq \hat{m}_i \leq m_i^{\text{max}} \), which can be rewritten using the anti-diffusive fluxes as
\[
m_i^{\text{max}} - \hat{m}_i \leq \sum_k C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k \leq m_i^{\text{max}} - \hat{m}_i.
\]
The sum in these constraints is split into two parts according to the sign of the anti-diffusive fluxes, so that the right-hand (max) inequality is written as
\[
\sum_k C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k \leq \sum_{\Delta \rho_i < 0} C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k
\]
Similarly, the sufficient condition for the lower (min) constraint is
\[
\sum_{\Delta \rho_i > 0} C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k \leq Q_i^{\text{min}}.
\]
In each cell, we define the incoming and outgoing total anti-dif
M \[\mathbf{dF}^{\text{em}}_k \geq 0, \mathbf{dF}^{\text{em}}_k \leq \mathbf{dF}^{\text{em}}_k \leq 0 \text{ and for nonzero } \mathbf{p}^\star, \text{ the auxiliary values } D_k^{\text{min}} = Q_i^{\text{min}} / \mathbf{dF}^{\text{em}}_k \geq 0, D_k^{\text{max}} - C_{\min}^{\text{FCR}} / \mathbf{dF}^{\text{em}}_k \geq 0. \]
For \( C_{\min}^{\text{FCR}} \leq \mathbf{dF}^{\text{em}}_k \leq 0 \text{ then the density bounds (2) are satisfied. At the interface } (i,k) \text{ we have to apply the sufficient conditions for bounds in both neighboring cells } i \text{ and } k \). So if for \( \mathbf{dF}^{\text{em}}_k \leq 0 \text{ we choose } C_{\min}^{\text{FCR}} = \min(D_k^{\text{max}}, D_k^{\text{max}}, 1) \text{ and for } \mathbf{dF}^{\text{em}}_k > 0 \text{ we choose } C_{\min}^{\text{FCR}} = \min(D_k^{\text{max}}, D_k^{\text{max}}, 1) \text{ then the density bounds (2) are satisfied for any } 0 \leq C_{\min}^{\text{FCR}} \leq C_{\min}^{\text{FCR}}. \text{ Note that if } \mathbf{dF}^{\text{em}}_k > 0 \text{ or } \mathbf{dF}^{\text{em}}_k = 0 \text{ the corresponding } D_k^{\text{em}} \text{ are not used and remain undefined.}

4. Synchronized FCR for density and momentum

Just as for the mass, we define the momentum anti-diffusive fluxes by \( \mathbf{dF}^{\text{em}}_k = F^{\text{em}H}_k - F^{\text{em}L}_k \). The FCR momentum flux is now given by \( F^{\text{em}H}_k = F^{\text{em}H}_k + \sum_k C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k \) and the new FCR momentum is, as for the mass,
\[
\hat{\mu}_i^{\text{FCR}} = \bar{\mu}_i + \sum_k F_k^{\text{em}F} = \mu_i^{\text{max}} + \sum_k C_{\min}^{\text{FCR}} \mathbf{dF}^{\text{em}}_k.
\]
Note that even when velocity, momentum and its fluxes are vectors we use the scalar limiting factor \( C_{\min}^{\text{FCR}} \) as we do not want to
affect the vectors’ directions by limiting. The local bounds on velocity are \( \mathbf{u}^{\text{min}} \leq \mathbf{u} \leq \mathbf{u}^{\text{max}} \), where the vector inequalities are taken component by component. Velocity is a derived quantity and the velocity bounds \( \mathbf{u}^{\text{min}} \leq \mu_k/\bar{m}_k \leq \mathbf{u}^{\text{max}} \) are rewritten as constraints on new momentum and mass \( \bar{m}_k \mathbf{u}^{\text{min}} \leq \mu_k \leq \bar{m}_k \mathbf{u}^{\text{max}} \) which, after insertion of fluxes, reads

\[
\left( \bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n \right) \mathbf{u}^{\text{min}} \leq \left( \bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n \right) \mathbf{u}^{\text{max}}.
\]

This is a coupled system of linear inequalities with respect to unknown correction factors \( C_{mk}^n \), \( C_{ik}^n \) at all interfaces (i,k). We want to stay as close as possible to high-order fluxes, i.e. choose this inequality and set

\[
\sum_k \left( C_{ik}^n \mathbf{F}_{ik}^n - \mathbf{u}^{\text{max}} C_{ik}^n \mathbf{F}_{ik}^m \right) \leq \mathbf{u}^{\text{max}} \bar{m}_k^2 - \bar{m}_k^2 \mathbf{u}^{\text{max}} \leq \mathbf{Q}_{i}^{\text{max}}. \tag{3}
\]

We denote the corrected contribution from the face (i,k) to the left-hand side of (3) by

\[
\Phi_{ik}^{\text{max}} = C_{ik}^n \mathbf{F}_{ik}^n - \mathbf{u}^{\text{max}} C_{ik}^n \mathbf{F}_{ik}^m \]

and the corresponding uncorrected quantity by

\[
\Psi_{ik}^{\text{max}} = \mathbf{F}_{ik}^n - \mathbf{u}^{\text{max}} \mathbf{F}_{ik}^m.
\]

so that inequality (3) now reads

\[
\sum_k \Phi_{ik}^{\text{max}} \leq \mathbf{Q}_{i}^{\text{max}}. \tag{4}
\]

Similarly as for density, we subdivide the sum into two parts according to the sign of \( \Psi_{ik}^{\text{max}} \)

\[
\sum_{k, \Psi_{ik}^{\text{max}} > 0} \Phi_{ik}^{\text{max}} + \sum_{k, \Psi_{ik}^{\text{max}} < 0} \Phi_{ik}^{\text{max}} \leq \mathbf{Q}_{i}^{\text{max}}.
\]

and the sufficient conditions for this inequality are

\[
\sum_{k, \Psi_{ik}^{\text{max}} > 0} \Phi_{ik}^{\text{max}} \leq 0, \quad \sum_{k, \Psi_{ik}^{\text{max}} < 0} \Phi_{ik}^{\text{max}} \leq \mathbf{Q}_{i}^{\text{max}}. \tag{5}
\]

The sufficient condition for the first resp. second inequality from (4)

\[
\Phi_{ik}^{\text{max}} \leq 0 \quad \text{for} \quad \Psi_{ik}^{\text{max}} \leq 0,
\]

\[
\Phi_{ik}^{\text{max}} \leq \mathbf{D}_{ik}^{\text{max}} \Psi_{ik}^{\text{max}} \quad \text{for} \quad \Psi_{ik}^{\text{max}} > 0,
\]

where \( \mathbf{D}_{ik}^{\text{max}} \) are \( \mathbf{Q}_{i}^{\text{max}} \) and \( \mathbf{P}_{i}^{\text{max}} \) (all inequalities, arithmetic operations and sums of vectors are taken component by component). The derived constraints are linear in the correction factors \( C_{ik}^n \), \( C_{ik}^n \) and local, i.e. they are constraints on the interface (i,k) which are sufficient for the maximum velocity bound in the cell i. Similar derivation for the minimum velocity bound gives another set of constraints. The complete set of constraints for interface (i,k) includes constraints derived from both max/min velocity bounds in both cells i and k sharing the interface (i,k); the constraints \( 0 \leq C_{ik}^n \leq C_{ik}^n \) derived in the previous section from density bounds and the constraints \( 0 \leq C_{ik}^n \leq 1 \). The solution of the complete system of linear inequalities for \( C_{ik}^n \) and \( C_{ik}^n \) is a non-empty \( (C_{ik}^n, C_{ik}^n) \) corresponding to the low-order remap always satisfies all the constraints) convex polygon in \( (C_{ik}^n, C_{ik}^n) \) space. Now the task is to find a pair \( (C_{ik}^n, C_{ik}^n) \) that is in some sense optimal, producing a remap close to the high-order one, i.e. with \( (C_{ik}^n, C_{ik}^n) \) close to (1,1). We minimize the functional

\[
f(C_{ik}^n, C_{ik}^n) = -C_{ik}^n \|\mathbf{dF}_{ik}^n\| \left( \frac{1}{\rho_k} + \frac{1}{\rho_k} \right) - C_{ik}^n \|\mathbf{dF}_{ik}^n\| \left( \frac{1}{\rho_k \mathbf{u}_k} + \epsilon + \frac{1}{\rho_k \mathbf{u}_k} + \epsilon \right), \tag{6}
\]

where \( \epsilon \) is a small quantity, under the derived constraints on \( (C_{ik}^n, C_{ik}^n) \). The functional is proportional to the local \( L_1 \) deviation of the SFCR remapped density and momentum from the high-order ones. This minimization is local, i.e. done for each interface (i,k) separately. The minimum is always in one of the polygon’s vertices, the polygon is intersection of eight half-planes (given by linear constraints), so the minimization in this case is easy. For more details on synchronized FCR for density and momentum see [13], where the choice of dimensional factors in a global deviation (the global analog of (6)) is made to produce a local minimization problem which is computationally efficient. Sensitivity to the choice of dimensional factors has not been investigated.

5. Synchronized FCR for density, momentum and energy

Just as for the mass and momentum we first define the total energy anti-diffusive fluxes by \( \mathbf{dF}_{ik}^n = \mathbf{F}_{ik}^n - \mathbf{F}_{ik}^m \). The FCR energy flux is now given by \( \mathbf{F}_{ik}^n, \mathbf{F}_{ik}^m \) and the new FCR total energy is

\[
E_{ik}^n = E_{ik} + \sum_k \mathbf{F}_{ik}^n \mathbf{dF}_{ik}^m = E_{ik} + \sum_k C_{ik}^n \mathbf{dF}_{ik}^m.
\]

The local bounds on specific internal energy are \( e_{ik}^{\text{min}} \leq e_{ik} \leq e_{ik}^{\text{max}} \). Similar derivation for the minimum velocity reads

\[
\sum_k \left( C_{ik}^n \mathbf{dF}_{ik}^n (\bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n) - C_{ik}^n \mathbf{dF}_{ik}^m \right) \leq e_{ik}^{\text{max}} (\bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n)^2. \tag{7}
\]

We rearrange this inequality to get all \( C \)'s to the left-hand side

\[
\sum_k \left( C_{ik}^n \mathbf{dF}_{ik}^n (\bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n) + \sum_k C_{ik}^n \mathbf{dF}_{ik}^m \right)
\]

\[
-\sum_k C_{ik}^n \mathbf{dF}_{ik}^m \left( \mathbf{F}_{ik}^n + \frac{1}{2} C_{ik}^n \mathbf{dF}_{ik}^n \right) + \frac{1}{2} \sum_k C_{ik}^n \mathbf{dF}_{ik}^n \mathbf{dF}_{ik}^m
\]

\[
+ \sum_k C_{ik}^n \mathbf{dF}_{ik}^m (\bar{m}_k^2 - e_{ik}^{\text{max}} (2 \bar{m}_k^2 + \sum_k C_{mk}^n \mathbf{F}_{ik}^n)) - e_{ik}^{\text{max}} \sum_k C_{ik}^n \mathbf{dF}_{ik}^m \mathbf{dF}_{ik}^m \}
\]

\[
\leq e_{ik}^{\text{max}} (\bar{m}_k^2)^2 - \bar{m}_k^2 (\bar{m}_k^2)^2 + \frac{1}{2} (\bar{m}_k^2)^2 = Q_{ik}^{\text{max}} \geq 0 \tag{8}
\]

To get a suitable sufficient condition for (8) we increase its left hand side (the part before \( \leq \)) by neglecting the negative (or positive as appropriate according to the sign) parts of the sums over \( i \) and setting \( C_{ik} = 1 \) in these sums. So the sufficient condition for (8) is
\[
\sum_k \left\{ C^k_i \left[ dF^e_i (m^i_l + C^m_i dF^m_i) + \sum_{l \neq k} dF^e_i dF^m_i \right] \right. \\
- C^k_i \left[ dF^m_i \left( \beta^i_l + 2 C^m_i dF^m_i \right) + \frac{1}{2} \sum_{l \neq k} dF^m_i dF^m_i \right] \\
+ C^m_i \left[ dF^m_i \left( E^i_l - \epsilon_1^i (2m^i_l + C^m_i dF^m_i)) \right) - \epsilon_2^i \sum_{l \neq k} dF^m_i dF^m_i \right) \right\} \leq Q_i^{E^{\text{max}}} 
\]

where
\[
\sum_l x_{il} = \sum_l x_{il}, \quad \sum_l x_{il} = \sum_l x_{il}. 
\]

We define corrected \( \psi_{ik}^{E^{\text{max}}} \) and uncorrected \( \psi_{ik}^{E^{\text{max}}} \) contribution from face \((i,k)\) to the left-hand side of (9) by
\[
\phi_{ik}^{E^{\text{max}}} = \psi_{ik}^{E^{\text{max}}} (1, 1, 1). 
\]

Similarly as for velocity constraints we split the sum in the energy constraint (9) into two parts according to the sign of \( \psi_{ik}^{E^{\text{max}}} \)
\[
\sum_{k, \psi_{ik}^{E^{\text{max}}} > 0} \psi_{ik}^{E^{\text{max}}} + \sum_{k, \psi_{ik}^{E^{\text{max}}} < 0} \psi_{ik}^{E^{\text{max}}} \leq Q_i^{E^{\text{max}}} 
\]
and the sufficient conditions for this inequality are
\[
\sum_{k, \psi_{ik}^{E^{\text{max}}} > 0} \psi_{ik}^{E^{\text{max}}} \leq 0, \quad \sum_{k, \psi_{ik}^{E^{\text{max}}} < 0} \psi_{ik}^{E^{\text{max}}} \leq Q_i^{E^{\text{max}}}.
\]

The sufficient conditions for the first resp. second inequality from (10) are
\[
\phi_{ik}^{E^{\text{max}}} \leq 0 \quad \text{for} \quad \psi_{ik}^{E^{\text{max}}} \leq 0, \\
\phi_{ik}^{E^{\text{max}}} \leq D^{E^{\text{max}}} - \psi_{ik}^{E^{\text{max}}} \quad \text{for} \quad \psi_{ik}^{E^{\text{max}}} > 0,
\]
where \(D^{E^{\text{max}}} = Q_i^{E^{\text{max}}} / \psi_{ik}^{E^{\text{max}}} \) and \(P_i^{E^{\text{max}}} = \sum_{k, \psi_{ik}^{E^{\text{max}}} > 0} \psi_{ik}^{E^{\text{max}}} \) (note that \(P_i^{E^{\text{max}}} \) is needed only when \(\psi_{ik}^{E^{\text{max}}} > 0\) which implies that \(P_i^{E^{\text{max}}} > 0\) and \(D^{E^{\text{max}}} \) is well defined). The sufficient conditions (11) for the maximum energy constraint (7) in the cell \(i\) are non-linear constraints in the correction factors \( \left( C_{ik}^m, C_{ik}^c, C_{ik}^{mL} \right) \) on the interface \((i,k)\). In a similar way one derives the sufficient conditions for the minimum energy constraint.

Let us summarize the constraints on the unknown correction factors \( \left( C_{ik}^m, C_{ik}^c, C_{ik}^{mL} \right) \) on the interface \((i,k)\). First, to stay between the low-order and high-order fluxes we require \(0 \leq C_{ik}^m \leq 1, \quad 0 \leq C_{ik}^c \leq 1, \quad 0 \leq C_{ik}^{mL} \leq 1\). Next, from the density bounds we derived in Section 3, the constraint \( C_{ik}^m \leq C_{ik}^{mL} \leq 1\). Further, we derived two (min/max) linear constraints (5) in \( \left( C_{ik}^m, C_{ik}^c \right) \) from the velocity bounds in cell \(i\) and we have to apply similar two (min/max) constraints from velocity bounds in cell \(k\). Finally we have to include two (min/max) non-linear constraints of type (11) in \( \left( C_{ik}^m, C_{ik}^c, C_{ik}^{mL} \right) \) from energy bounds in cell \(i\) and also two more such constraints from energy bounds in cell \(k\). Remembering that we want to stay close to the high-order scheme (i.e. maximize \(C_s\)) we first try if \( \left( C_{ik}^m, C_{ik}^c, C_{ik}^{mL} \right) = (C_{ik}^{\text{high}}, 1, 1) \) satisfies all constraints and if it does we use on the interface \((i,k)\) these values (which is probably the case in smooth regions where we avoid optimization). Otherwise we minimize the local deviation from the high-order fluxes given by the functional
\[
f(C_{ik}^m, C_{ik}^c, C_{ik}^{mL}) = -C_{ik}^m \| dF^m_i \| \left( \frac{1}{\rho_i^1} + \frac{1}{\rho_i^2} \right) \\
- C_{ik}^c \| dF^e_i \| \left( \frac{1}{|n_i| + \epsilon} + \frac{1}{|e_i| + \epsilon} \right) \\
- C_{ik}^{mL} \| dF^m_i \| \left( \frac{1}{e_i^1 + \epsilon} \right), 
\]
where \(n_i = \rho_i u_i\) and \(e_i = \rho_i (u_i + u_i^2/2)\), under all derived constraints on \( \left( C_{ik}^m, C_{ik}^c, C_{ik}^{mL} \right) \). So on each interface \((i,k)\) we get one constrained minimization problem which gives us the SFCR correction factors.

6. Numerical tests

In this section we present several numerical tests showing the performance of the developed synchronized FC for density, momentum and energy. For the tests we use the 1D cyclic remapping [2] defined by the sequence of meshes generated using the analytical function
\[
x(\xi, t) = x_{\text{min}} + (x_{\text{max}} - x_{\text{min}}) \tilde{x}(\xi, t), \quad \tilde{x}(\xi, t) = \left[ 1 - \alpha(t) \right] \xi + \alpha(t) t^2, \\
\alpha(t) = \frac{\sin(4\pi t)}{2}, \quad 0 \leq \xi \leq 1, \quad 0 \leq t \leq 1.
\]

The mesh node positions \( \{x_i^k\} \) are defined as \( x_i^k = x(\xi_i^k, t^k) = x((i-1)/N, k/k_{\text{max}}) \) with \( k = 0, \ldots, k_{\text{max}} \) corresponding to pseudo-time steps and \( i = 1, \ldots, N + 1 \) corresponding to spatial node positions. Here \( N \) is the number of spatial cells and \( k_{\text{max}} \) is the total number of pseudo-time steps. The rate of convergence is tested by refining the resolution \( (N, k_{\text{max}}) = (64, 320), \quad (128, 640), \)

![Fig. 1. Simple shock with 64 cells after 320 remaps by high-order, low-order and SFCR schemes.](image-url)
The given variables \((q, u, e)\) are initially defined on the initial uniform mesh. In each pseudo-time step the mesh at \(t_k\) is moved to the new position given by the new pseudo-time \(t_{k+1}\) and the variables are remapped to the new mesh. The last mesh at pseudo-time \(t_{k_{\text{max}}}\) is the same as the first initial uniform mesh, so the exact solution is given by the initial profile and we can compute errors of approximate remap and its convergence rate.

### 6.1. Simple shock

The first test is a simple shock connecting two constant states. Initial (and exact final) density, velocity and internal energy profiles are given by

\[
\begin{align*}
(q, u, e)(x) &= \left\{
\begin{array}{ll}
4, 1.115/126 & \text{for } 0 \leq x < 0.5, \\
1, 0.20/63 & \text{for } 0.5 \leq x \leq 1.
\end{array}
\right.
\end{align*}
\]

The final result after 320 cyclic remaps on the mesh (12) with 64 cells is shown in Fig. 1 computed by the low-order, high-order and SFCR schemes. Clearly the low-order scheme is too diffusive, the high-order scheme produces overshoots and undershoots while SFCR scheme does produce neither overshoots nor undershoots while keeping the same gradient at the shock as the high-order scheme. The table of convergence is presented in Table 1 which also includes results by Schär and Smolarkiewicz (S&S) method [11] for density and velocity (S&S paper does not deal with internal energy bounds). For the description of S&S remapping and

### Table 1

Convergence tables \((L_1\) errors and their ratios) in density, velocity and internal energy for cyclic remapping of simple shock by low-order, high-order, SFCR and S&S methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Resolution</th>
<th>(N)</th>
<th>(k_{\text{max}})</th>
<th>(L_1) (\epsilon)</th>
<th>(\rho)</th>
<th>(u)</th>
<th>(e)</th>
<th>(\text{Err. prev./curr. resol.})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-order</td>
<td>64</td>
<td>320</td>
<td>0.121</td>
<td>0.262</td>
<td>0.158</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
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<tr>
<td>128</td>
<td>640</td>
<td>0.086</td>
<td>0.186</td>
<td>0.112</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>1280</td>
<td>0.061</td>
<td>0.131</td>
<td>0.080</td>
<td>1.4</td>
<td>1.4</td>
<td>1.4</td>
<td></td>
</tr>
<tr>
<td>High-order</td>
<td>64</td>
<td>320</td>
<td>0.038</td>
<td>0.093</td>
<td>0.064</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
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<tr>
<td>128</td>
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<td>0.055</td>
<td>0.038</td>
<td>1.7</td>
<td>1.7</td>
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<td></td>
</tr>
<tr>
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<td>0.033</td>
<td>0.023</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>SFCR</td>
<td>64</td>
<td>320</td>
<td>0.030</td>
<td>0.062</td>
<td>0.036</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
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<tr>
<td>256</td>
<td>1280</td>
<td>0.011</td>
<td>0.022</td>
<td>0.013</td>
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<td>1.7</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>S&amp;S</td>
<td>64</td>
<td>320</td>
<td>0.043</td>
<td>0.078</td>
<td>–</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>128</td>
<td>640</td>
<td>0.023</td>
<td>0.045</td>
<td>–</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>1280</td>
<td>0.015</td>
<td>0.032</td>
<td>–</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
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</tr>
</tbody>
</table>

### Table 2

Convergence tables \((L_1\) errors and their ratios) in density, velocity and internal energy for cyclic remapping of exponential shock by low-order, high-order, SFCR and S&S methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Resolution</th>
<th>(N)</th>
<th>(k_{\text{max}})</th>
<th>(L_1) (\epsilon)</th>
<th>(\rho)</th>
<th>(u)</th>
<th>(e)</th>
<th>(\text{Err. prev./curr. resol.})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low-order</td>
<td>64</td>
<td>320</td>
<td>0.257</td>
<td>0.341</td>
<td>0.119</td>
<td>1.4</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>128</td>
<td>640</td>
<td>0.190</td>
<td>0.205</td>
<td>0.068</td>
<td>1.4</td>
<td>1.7</td>
<td>1.7</td>
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<tr>
<td>256</td>
<td>1280</td>
<td>0.163</td>
<td>0.128</td>
<td>0.040</td>
<td>1.2</td>
<td>1.6</td>
<td>1.7</td>
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<tr>
<td>High-order</td>
<td>64</td>
<td>320</td>
<td>0.122</td>
<td>0.056</td>
<td>0.014</td>
<td>1.8</td>
<td>1.9</td>
<td>1.7</td>
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<tr>
<td>128</td>
<td>640</td>
<td>0.068</td>
<td>0.030</td>
<td>0.008</td>
<td>1.5</td>
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<td>1.6</td>
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<tr>
<td>256</td>
<td>1280</td>
<td>0.046</td>
<td>0.018</td>
<td>0.005</td>
<td>1.5</td>
<td>1.7</td>
<td>1.6</td>
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<tr>
<td>SFCR</td>
<td>64</td>
<td>320</td>
<td>0.151</td>
<td>0.049</td>
<td>0.012</td>
<td>1.9</td>
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<td>0.025</td>
<td>0.006</td>
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<tr>
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<td>1280</td>
<td>0.047</td>
<td>0.014</td>
<td>0.004</td>
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<tr>
<td>S&amp;S</td>
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<td>320</td>
<td>0.213</td>
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<td>–</td>
<td>1.6</td>
<td>2.1</td>
<td>–</td>
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<td>128</td>
<td>640</td>
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<td>0.039</td>
<td>–</td>
<td>1.4</td>
<td>1.8</td>
<td>–</td>
<td></td>
</tr>
<tr>
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<td>1280</td>
<td>0.095</td>
<td>0.022</td>
<td>–</td>
<td>1.4</td>
<td>1.8</td>
<td>–</td>
<td></td>
</tr>
</tbody>
</table>
its comparison with SFCR see [13]. The SFCR method produces smaller errors than both high-order and S&S methods. The SFCR order of convergence is the same as that for the high-order method.

6.2. Exponential shock

The second test is an exponential shock with initial (and exact final) density, velocity and internal energy profiles given by

\[
(p, u, e)(x) = \begin{cases} 
3\rho_0 \exp \left( \frac{2x-3}{4} \right) \left( 1 + 2\eta \right) \frac{1}{\eta} \left( \frac{1}{2\eta} \right)^{1/2} & \text{for } 0 \leq x \leq x_f, \\
(\rho_0 \exp \left( \frac{-3x}{4} \right), 0, 0) & \text{for } x_f < x \leq 15, 
\end{cases}
\]

where \( x_f = x_0 + \frac{1}{2} \delta \log (t/t_0), \quad \eta = \frac{x-x_0}{t}, \quad t_0 = 2, \quad x_0 = 6, \quad \rho_0 = 1, \quad \delta = 4, \quad t = 6. \) The final result after 320 cyclic remaps on the mesh (12) with 64 cells is shown in Fig. 2 computed by the low-order, high-order and SFCR schemes. The low-order scheme smears the peak in density completely, the high-order scheme produces under-shoots in velocity and internal energy while SFCR is free of under-shoots and overshoots. The table of convergence is presented in Table 2 which also includes results by the S&S method [11]. SFCR is better than S&S, it is also better than the high-order method in velocity and internal energy and only slightly worse than the high-order method for density (SFCR convergence is better even for density). The SFCR order of convergence is about the same as that for the high-order method.

7. Conclusion

We have developed a new optimization-based synchronized flux corrected remapping of mass/density, momentum/velocity and specific internal energy to be used in the ALE methods. The main new results are derivation of a local linear objective function and new sufficient conditions for preservation of bounds on internal energy.

Acknowledgments

This work was performed under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory under Contract DE-AC52-06NA25396. The authors gratefully acknowledge the partial support of the US DOE NNSA’s Advanced Simulation and Computing (ASC) Program and the partial support of the US DOE Office of Science Advanced Scientific Computing Research (ASCR) Program in Applied Mathematics Research. The first and the third author have been supported in part by the Czech Ministry of Education grants MSM 6840770022 and LC528 and are grateful for support from the European Science Foundation OPTPDE Network on Optimization with PDE Constraints.

The authors thank P. Smolarkiewicz, D. Kuzmin, C. Schär, B. Kashiwa, T. Ringler, J. Shadid, L. Margolin and B. Rider for fruitful discussions and also anonymous reviewers for constructive comments.

References