APPLICATIONS OF QUANTIFIER ELIMINATION THEORY TO CONTROL SYSTEM DESIGN

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Abstract

In this paper we show how a number of interesting linear control system analysis and design problems can be reduced to Quantifier Elimination (QE) problems. We assume a fixed structure for the compensator, with design parameters q_i. The problems considered are problems that currently have no general solution. However, the problems must be of modest complexity if existing QE software packages are to produce answers. The software package QEPCAD is used to solve some numerical design examples.

1 Introduction

As will be demonstrated in this paper, many interesting control system design and analysis problems can be reduced to *Quantifier Elimination* or QE problems. In particular, for linear time-invariant systems, important control issues such as *stability, robust stability*, and *robust performance* can be reduced to systems of multivariable-polynomial inequalities with logic quantifiers such as "there exits" ($\exists$) and "for all" ($\forall$). Typically the variables in the polynomials are real and are related to plant (controlled system) and compensator (controller) parameters. The final design objective is to obtain quantifier-free formulas for the compensator parameters or, for the existence problem, to obtain a "true" or "false" output. QE methods are especially attractive for control problems where there are no general analytical design algorithms and where, for practical reasons, one would like to have the simplest possible compensator. An example of this is the static *output-feedback stabilization problem* which is the problem of finding a "zero-order" compensator, or what is commonly referred to *simple proportional feedback*, such that the closed-loop system is stabilized. Proportional feedback is the simplest possible type of feedback that can be used, yet this problem remains an open analytical problem. Indeed one of the first attempts to use QE methods, also referred to as *Decision Methods* to solve control system design problems, was to solve the output-feedback stabilization problem, as reported in the paper by Anderson et. al. [1]. Unfortunately, the QE algorithms at the time were very complex and no software was available for computer solutions. Since then, improved QE algorithms have been developed, see for example [2, 3], and software packages have been written to implement the new algorithms, for example the software package *Quantifier Elimination by Partial Cylindrical Algebraic Decomposition* package (QEPCAD) [4, 5].

This paper is organized as follows. Section 2 contains a discussion of the control problems we studied using QE. It is divided into stabilization problems and performance problems. Section 3 contains an overview of the mathematical theory of QE, with special emphasis on issues arising in control problems. Examples of using QE to solve control problems are given in section 4. Our conclusions and directions for future research are included in section 5.

2 Control Design Problems

In this section, we review a few control problems divided into two categories: The first contains problems where the main objective is to stabilize one or a family of systems. The second category contains in addition to stabilizability, some performance requirements. In the sequel, we refer to the standard feedback block diagram used in control theory. The functions $C(s,q)$ and $G(s,p)$ are the Laplace-transform transfer functions of the compensator and the plant, respectively. We assume here that our system is linear, time-invariant, and lumped, so that these two functions are rational functions in the transform variable s. The plant may also be characterized in state-space form, i.e.

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x \tag{2.1}$$

with the usual relation $G(s,p) = C(p)[sI - A(p)]^{-1}B(p)$.

2.1 Stability

For feedback control systems, the first design problem one must address is that of stability of the closed-loop system. Stability requires that all the zeros of the numerator polynomial (closed-loop characteristic polynomial) of the rational function $1+C(s,q)G(s,p)$ have *negative real parts*. If the closed loop characteristic polynomial is denoted $f(s) = a_0 s^n + a_1 s^{n-1} + ... + a_n, \quad a_0 > 0$, the Liénard-Chipart criterion [6] then states that all the roots of $f(s) = 0$ have negative real parts if and only if $a_n > 0, a_{n-2} > 0, \cdots; \Delta_1 > 0, \Delta_3 > 0, \cdots$ where Δ_i is the Hurwitz determinant of order i, i.e.

$$\begin{aligned} \Delta_i &= \det \begin{bmatrix} a_1 & a_3 & a_5 & \cdots \\ a_0 & a_2 & a_4 & \cdots \\ 0 & a_1 & a_3 & \cdots \\ 0 & a_0 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & a_i \end{bmatrix}; \\ a_k &= 0; \quad k > n \end{aligned} \tag{2.2}$$

See section 13, chapter XV of reference [6] for more details. We list next four stabilizability problems which will later be framed as QE problems, and solved using the software package QEPCAD.

Problem 1: Output Feedback Stabilization The output feedback problem is one of the most important open question in control engineering. The problem is stated as follows: Given the linear, time-invariant system (2.1), find a static output feedback $u(t) = -Ky(t)$ so that the closed-loop system has some desirable characteristics, or determine that such a feedback does not exist. The problem is important in its own right, but also because many other problems, e.g. fixed-order compensation, are reducible to some variation of it. For multi-input-multi-output (MIMO) systems, this problem has no analytical solution, even when all the plant parameter values are known.

Problem 2: Simultaneous Stabilization The problem of stabilizing n different plants is a long-standing problem in the robust control literature. The problem is stated as one of finding conditions for the existence and a method of designing one controller to stabilize a set of n plants: G_1, G_2, ..., G_n, or determining that such a controller can not exist. The controller should be exhibited once its existence conditions are satisfied. The problem is relevant in applications where the plant is only known to belong to a set of n different plants, or where the failure of sensors or actuators will drastically change the plant from its current description. The problem has no analytical solution if $n \geq 2$.

Problem 3: Robust Stabilization In many situations, the mathematical description of a physical plant is not exactly known. In special but important situations, the uncertainty of the plant is in the values of its parameters, which are assumed to appear as polynomials in the coefficients of s in the plant transfer function $G(s,p)$ or as polynomial functions in the entries of the A, B, and C matrices. For example

$$G(s,p) = \frac{a_{n-1}(p)s^{n-1} + \cdots + a_0(p)}{b_n(p)s^n + \cdots + b_0(p)}$$

where the a_i, b_i are polynomials in the entries of the vector p, or $A(p) = A + \alpha_1(p)A_1 + ... + \alpha_n(p)A_n$ where A_i are known matrices, and $\alpha_i(p)$ are polynomial in the entries of the vector p. The robust stabilization problem is to determine if a compensator vector q exists which stabilizes the closed-loop system, given the uncertainty in plant vector p; and when one does exist to compute a value for q, or characterize the set of vectors q that preserve stability. It has been shown that for the transfer function model, using Kharitonov theory, that some problems of this type can be reduced to the problem of simultaneous stabilization of a finite collection of plants [7, 8].

2.2 Performance

In general, stability is but one of many requirements of a closed-loop control system. We will focus here on performance measures specified in the frequency domain, since measures of this type fit the QE problem formulation very nicely.

Problem 4: Frequency Domain Multiobjective Design With respect to the feedback system, some typical performance objectives to be met (simultaneously) are represented by the inequalities

- **Tracking Error** $|S(j\omega)| \leq \delta, \;\; 0 < \omega \leq \omega_1$
- **Control Effort** $|W(j\omega)| < \epsilon, \;\; \text{for all}\, \omega$

where the function $S(s)$ represents the transfer function between the command signal r and the error signal e, and the function $W(s)$ represents the transfer function between the command signal r and the control signal u. These two functions are given by

$$S(s) = \frac{1}{1 + C(s,q)G(s,p)}, \;\; W(s) = \frac{C(s,q)}{1 + C(s,q)G(s,q)} \tag{2.3}$$

By computing the magnitude squared and clearing fractions, these two specifications can be reduced to polynomial inequalities in the variables q_i, p_i, and ω, with integer coefficients if δ^2 and ϵ^2 are rational numbers. For *robust* performance the various performance specifications must be met for all admissible plant vectors p. For *nominal* performance, the specification must be met only for a fixed vector p. For QE to apply, the entries of the vector p must be rational numbers.

In the next section, we review the QE problem as it relates to control, and discuss the availability of software packages for the purpose of solving QE problems.

3 QE Methods and QEPCAD Software

In this section, we review the general QE problem and introduce the software package QEPCAD which we use to solve our control problems. A more detailed treatment may be found in [9, 10].

Given the set of polynomials with integer coefficients $P_i(X,Y), \;\; 1 \leq i \leq s$ where X represents a k dimensional vector of quantified real variables and Y represents a l dimensional vector of unquantified real variables, let $X^{[i]}$ be a block of k_i quantified variables, Q_i be one of the quantifiers $\exists$ (there exists) or $\forall$ (for all), and let $\Phi(Y)$ be the quantified formula

$$\Phi(Y) = (Q_1 X^{[1]}, ..., Q_w X^{[w]}) F(P_1, ..., P_s), \tag{3.4}$$

where $F(P_1, ..., P_s)$ is a quantifier free *Boolean* formula, that is a formula containing the Boolean operators $\wedge$ (and) and $\vee$ (or), operating on *atomic predicates* of the form $P_i(Y, X^{[1]}, ..., X^{[w]}) \geq 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) > 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) = 0$. We can now state the general quantifier elimination problem

General Quantifier Elimination Problem: Find a quantifier-free Boolean formula $\Psi(Y)$ such that $\Phi(Y)$ is true if and only if $\Psi(Y)$ is true. In control problems, the unquantified variables are generally the compensator parameters, represented by the parameter vector $Y = q$, and the quantified variables are the plant parameters, represented by the plant parameter vector p, and the frequency variable ω. Uncertainty in plant parameters are characterized by quantified formulas of the type $\forall(p_i) \;\; [\underline{p_i} \leq p_i \leq \overline{p_i}]$ where $\underline{p_i}$ and $\overline{p_i}$ are rational numbers. *The quantifier-free formula $\Psi(q)$ then represents a characterization of the compensator design.*

An important special problem is the QE problem with no unquantified variables (free variables), i.e. $l = 0$. This problem is referred to as the *General Decision Problem*.

General Decision Problem: With no unquantified variables, i.e. $l = 0$, determine if the quantified formula given in 3.4 is true or false.

The general decision problem may be applied to the problem of *existence* of compensators that meet given specifications, in which case an "existence" quantifier is applied to the compensator parameter q.

Algorithms for solving general QE problems were first given by Tarski [9] and Seidenberg [11], and are commonly called Seidenberg-Tarski decision procedures. Tarski showed that QE is solvable, but his algorithm and later modifications are exponential in the size of the problem. Researchers in Control Theory have been aware of Tarski's results and their applicability to Control problems since the 1970's but the tedious operations made the technique very limited [1]. Later, Collins [2] introduced a theoretically more efficient QE algorithm that uses a cylindrical algebraic decomposition (CAD) approach. However, this algorithm was not capable of effectively handling nontrivial problems. More recently Hong [4], Collins and Hong [3], Hong [5] have introduced a significantly more efficient partial CAD QE algorithm. In our work we use the Hong's implementation of the CAD algorithm called QEPCAD. The CAD algorithm always completely solves any QE problem. However, the computational cost is extremely high. Our experience (see [12]) indicates that QEPCAD can always solve, in a few seconds on a large workstation, most textbook examples. It can also solve some significantly harder problems and a few non-trivial problems. It is therefore important to simplify the QE problem as much as possible before using QEPCAD.

4 Control System Design Examples

The controller $C(s,q)$ is designed so that the closed-loop system satisfies some design requirements in the face of significant uncertainties in the description of $G(s,p)$. The examples in this subsection illustrate the application of QE methods to stability problems.

4.1 Stability

As noted in the introduction, stability is for linear time-invariant systems is generally checked by a criterion such as the Liénard-Chipart criterion, applied to a closed-loop characteristic polynomial.

: Output Feedback Stabilization Example Consider the static output feedback example in Anderson et.al. [1] where the Tarski-Seidenberg theory was applied manually. We have the plant

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \\ y &= \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x \end{aligned} \tag{4.5}$$

with the static output feedback, $u = -Kx$ where $K = [q_1, q_2] = [v, w]$ so that the closed-loop system matrix is

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w & 13+5v-w & -v \end{bmatrix} \tag{4.6}$$

with a closed-loop characteristic polynomial $s^3 + vs^2 + (w - 5v - 13)s + w$. The Liénard-Chipart criterion gives us the conditions

$$v > 0, \quad w > 0, \quad -5v^2 - 13v + vw - w > 0 \tag{4.7}$$

for the polynomial to be stable, i.e. to have all roots with negative real part. The solution of the inequalities (4.7) can be stated as the quantifier elimination problem. We have stated it as

$$\exists w,\ (v > 0) \wedge (w > 0) \wedge (-5v^2 - 13v + vw - w > 0).$$

The QEPCAD code eliminated w and gave us the unquantified equivalent formula $v - 1 > 0 \wedge v > 0$ in 0.034 seconds. Now we can choose $v > 1$, e.g $v_0 = 2$, and the inequalities (4.7) give us restrictions on w, namely for v_0, we obtain that $w > 0 \wedge w - 46 > 0$ so that one parameterization of stabilizing controllers is, $v_0 = 2, w > 46$.

Simultaneous Stabilization Example The problem here is to find a (stable) compensator which simultaneously stabilize three different plants with transfer functions

$$\begin{aligned} G_1(s) &= \frac{2-s}{(s^2-1)(s+2)}, \quad G_2(s) = \frac{2-s}{s^2(s+2)}, \\ G_3(s) &= \frac{2-s}{(s^2+1)(s+2)}. \end{aligned}$$

This problem is taken form [13] where it is shown, using certain sufficient conditions, that the problem can be solved using a third-order compensator. The problem is thus to explore first and second order (stable) compensators $C(s)$ such that the numerators of all the transfer functions $Q_i = 1 + CP_i,\ , i = 1, 2, 3$ are stable.

First-order compensator: As a first attempt we have chosen the stable compensator of the form

$$C = \frac{as+b}{s+d} \tag{4.8}$$

with $d > 0$. The result given by QEPCAD was "false" which means that there does not exist any real a, b, d for which guarantees closed-loop stability.

Second-order compensator with three parameters: In order to reduce the number of parameters to be considered we assume a second-order compensator of the form

$$C(s) = \frac{A(s+B)^2}{(s+D)^2} \tag{4.9}$$

with $D > 0$, for stability of the compensator. The Liénard-Chipart criterion gives us after some inequality simplification $A > 0, B > 0$ and

$$\begin{aligned} P_1 &> 0 \wedge P_2 > 0 \wedge P_3 > 0 \wedge P_4 P_5 > 0 \wedge A > 0 \\ &\wedge B P_6 P_7 > 0 \wedge P_8 P_9 > 0 \end{aligned}$$

where

$$\begin{aligned} P_1 &= AB^2 - D^2, \\ P_2 &= -AB + A + D^2 - D - 1, \\ P_3 &= AB - AD - 2A + D^3 + 4D^2 + 4D, \\ P_4 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 \\ &\quad +5D + 2, \\ P_5 &= AB^3 - AB^2D - 4AB^2 + 2ABD + 4AB \\ &\quad +2BD^3 + 5BD^2 + 2BD - D^3 - 4D^2 - 4D, \\ P_6 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 + 4D, \\ P_7 &= AB^2 - ABD - 4AB + 2AD + 4A + 2D^3 \\ &\quad +4D^2, \\ P_8 &= AB - 2A - BD^2 - 4BD - 4B + 2D^2 \\ &\quad +3D - 2, \\ P_9 &= AB^3 - AB^2D - 4AB^2 + 2ABD + 4AB \\ &\quad +2BD^3 + 3BD^2 - 2BD + D^3 + 4D^2 + 4D. \end{aligned} \tag{4.10}$$

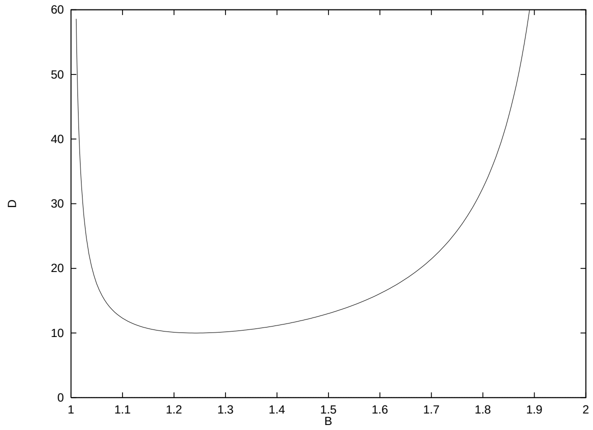

Figure 4.1: The set of acceptable values of B, D

QEPCAD was able to solve the existence problem, but only after 2 hours of CPU time. The QEPCAD output indicated that a solution did indeed exist, so that a second order stable stabilizing compensator does exist. To find a particular compensator and to reduce the CPU time require to find a solution, other supplementary techniques were used [14, 15], and further simplified manually. The QEPCAD package finally solved the quantifier elimination problem $\exists D$ with the result $1 < B < 2$. We chose $B = 3/2$, $D = 15$ (any $D > 13.01$ would work fine). These values of B, D resulted in $100 < A < 119$ from which we chose $A = 110$. Thus a particular second-order compensator is given by $A = 110, B = 3/2, D = 15$. It can be shown that the quantifier-free formula is equivalent to the points above the curve in Figure (4.1).

4.2 Performance

As noted in section 2, multiple frequency domain design objectives involving constraints on magnitude of transfer functions, can be reduced to QE problems with the $\forall$ quantifier on the the frequency variable ω. The following example illustrates a problem of this type.

Robust Multiobjective Design Example The problem considered here is a simplified version of the problem in [16]. The plant is assumed to be an unstable first order system with transfer function

$$G(s,p) = \frac{p_1}{1 - s/p_2}, \quad 0.8 \le p_1 \le 1.25 \qquad (4.11)$$

with simple output feedback $C(s,q) = q_1$. The tracking error bound (See section 2) is assumed to be given by $\delta = 0.1$, with $\omega_1 = 10$, and the control effort bound is given to be $\epsilon = 20$. To solve the robust multiobjective problem we must satisfy the following inequalities for *all* $16 \le 20p_i \le 25$. Note that the constraints on p_i have been re-written in terms of integers to satisfy the requirements of the QE theory, i.e. all numerical polynomial coefficients must be integers.

Robust stability: The stability criterion for the first order closed-loop characteristic polynomial is simply, $p_2(1 + p_1 q_1) < 0$

Tracking error: If the magnitude-squared of $S(j\omega)$ is computed, and the denominator polynomial is cleared we obtain the condition, $99\omega^2 + (p_2)^2(100(1+p_1q_1)^2 - 1) > 0, \ \ 0 \le \omega \le 10$

Control effort: With the same computations as for the tracking error, we obtain for control effort the condition, $(400 - q_1^2)\omega^2 + (p_2)^2(400(1+p_1q_1)^2 - q_1^2)$, all real ω. The direct solution from QEPCAD, with no preliminary simplifications, yields the quantifier-free formula

$$\begin{aligned} \Psi(q_1) \quad = \quad & [(q_1 + 20 \ge 0) \wedge (q_1 + 2 \le 0] \\ & \vee[(8q_1 + 11 < 0) \wedge (q_1 + 2 \ge 0)] \end{aligned} \qquad (4.12)$$

From (4.12) one obtains the following parameterization of output-feedback compensation which satisfies the robust multiobjective problem.

$$-20 \le q_1 < -1.375$$

5 Conclusions

Quantifier-elimination theory provides a method for the solution of control system design problems using simple fixed-structure compensators. While the currently available software limits the application of QE theory to modest size problems, the theory is valuable for the following reasons:

- One can solve problems for which there is no general analytical theory.
- From a practical point of view it is important to keep the compensator as simple as possible, since some on-line tuning of compensator parameters is almost always required.
- Even modest size practical problems cannot be effectively computed by hand because of the large number of inequalities involved.

QE methods also have the advantage of leaving no "holes" in the parameter space in the deterministic discretization case, or of yielding "probability-one" results in the random discretization case.

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