Zákony zachování

Integrální a diferenciální tvar

- 1D trubice naplněná ideálním plynem o hustotě $\rho(x,t)$ a vychlosti $v(x,t)$
- celková hmotnost v úseku (x_1,x_2)
$$\int_{x_1}^{x_2} \rho(x,t)\,dx$$
- tok hmotnosti bodem x
$$\rho(x,t)\,v(x,t)$$
- časová změna hmotnosti = rozdílu toků krajních bodech, Integrální tvar zákona zachování
$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)\,v(x_1,t) - \rho(x_2,t)\,v(x_2,t)$$
- integrací přes (t_1,t_2) dostaneme druhý tvar integrálního zákona zachování
$$(1)\quad \int_{x_1}^{x_2} \rho(x,t_2)\,dx - \int_{x_1}^{x_2} \rho(x,t_1)\,dx = \int_{t_1}^{t_2} \rho(x_1,t)\,v(x_1,t)\,dt - \int_{t_1}^{t_2} \rho(x_2,t)\,v(x_2,t)\,dt$$
- pro diferencovatelné $\rho(x,t)$, $v(x,t)$ platí
$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt$$
$$\rho(x_2,t)\,v(x_2,t) - \rho(x_1,t)\,v(x_1,t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}\big(\rho(x,t)\,v(x,t)\big)\,dx$$
- čili (1) lze přepsat
$$(2)\quad \int_{t_1}^{t_2}\int_{x_1}^{x_2} \left[\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\big(\rho(x,t)\,v(x,t)\big)\right] dx\,dt = 0$$
- (2) musí platit pro každý interval (t_1,t_2) a každý interval (x_1,x_2)
  => diferenciální tvar zákona zachování
$$\rho_t + (\rho v)_x = 0$$

Obecný tvar zákona zachování v 1D (konzervativní tvar)
$$\vec{U}_t + \left(\vec{F}(\vec{U})\right)_x = 0$$
$$= \vec{S}(\vec{U})$$

- skalární zákon zachování – konzervativní tvar

$$u_t + f(u)_x = 0$$

- advekční tvar

$$u_t + f_u \, u_x = 0$$ ---- rychlost šíření vln (f_u)

advekční rovnice

$$u_t + a \, u_x = 0$$

- advekční tvar pro systém, konzerv. tvar $\vec{U}_t + \vec{F}(\vec{U})_x = 0$

$$\vec{U}_t + \vec{F}_{\vec{U}} \cdot \vec{U}_x = 0$$

$$\vec{F}_{\vec{U}} = \text{Jakobián} = \begin{pmatrix} F^1_{u^1}, & F^1_{u^2}, & \dots \\ F^2_{u^1}, & F^2_{u^2}, & \dots \\ \vdots & & \end{pmatrix}$$

- zákony zachování jsou ~~hyper~~ hyperbolické

$\Rightarrow \vec{F}_{\vec{U}}(x,t)$ má reálná vlastní čísla $\forall_{x,t}$

- vlastní čísla

$$\det(\vec{F}_{\vec{U}} - \lambda_i \mathbb{I}) = 0 \quad , \ \lambda_i \in R, \ i = 1, \dots, n, \ \vec{U} \in R^n, \ \vec{F} \in R^n$$

- vlastní vektory

$$\vec{V}_i \cdot \vec{F}_{\vec{U}} = \lambda_i \vec{V}_i \qquad (\vec{V}_i \text{ jsou řádky})$$

- systém je striktně hyperbolický

$\Leftrightarrow \vec{F}_{\vec{U}}(x,t)$ má navzájem různá vlastní čísla a nezávislé vlastní vektory

- pro hyperbolický systém

$$\mathbb{P} = \begin{pmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vdots \\ \vec{V}_n \end{pmatrix}$$ je nesingulární matice

$$\mathbb{P} \cdot \vec{F}_{\vec{U}} = \underbrace{\begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}_{\Lambda} \cdot \mathbb{P}$$

- čili $\mathbb{P} \cdot \vec{F}_{\vec{U}} \cdot \mathbb{P}^{-1} = \Lambda$

- systém

$$\vec{U}_t + \mathbb{F}_{\vec{u}} \cdot \vec{U}_x = 0$$

$$\mathbb{P} \cdot \vec{U}_t + \mathbb{P} \cdot \mathbb{F}_{\vec{u}} \cdot \vec{U}_x = 0$$

systém lokálně v (x,t)
předpoklad zamrzlých koef.,
$\mathbb{P}$ nezávisí na (x,t)

$$(\mathbb{P} \cdot \vec{U})_t + \mathbb{P} \cdot \mathbb{F}_{\vec{u}} \cdot \mathbb{P}^{-1} \cdot \mathbb{P} \cdot \vec{U}_x = 0$$

- charakteristický systém

$$\vec{W}_t + \Lambda \cdot \vec{W}_x = 0 \implies W^i_t + \lambda_i W^i_x = 0 \quad i = 1, \dots, 4$$

- pro charakteristické proměnné

$$\vec{W} = \mathbb{P} \cdot \vec{U}$$

- lokální rozklad řešení do systému vlastních vektorů jakobiánu $\mathbb{F}_{\vec{u}}$
- vlastní čísla λ_i jsou rychlosti šíření vln

Slabé řešení

- testovací funkce $\varphi(x,t) \in C^1_0$ s kompaktním nosičem

$$u_t + f(u)_x = 0$$

$$\varphi u_t + \varphi f(u)_x = 0$$

$$\int_0^{\infty} \int_{-\infty}^{+\infty} [\varphi u_t + \varphi f(u)_x] \, dx \, dt = 0$$

- per partes + kompaktní nosič

$$\int_0^{\infty} \int_{-\infty}^{+\infty} [\varphi_t u + \varphi_x f(u)] \, dx \, dt = -\int_{-\infty}^{+\infty} \varphi(x,0) u(x,0) \, dx$$

- pokud platí $\forall \varphi \in C^1_0 (R \times R^+)$ pak u je slabým řešením zákona zachování

- integrální tvar $(x,t) \in (x_1, x_2) \times (t_1, t_2)$

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} (u_t + f(u)_x) \, dx \, dt = 0$$

$$\int_{x_1}^{x_2} (u(x,t_2) - u(x,t_1)) \, dx + \int_{t_1}^{t_2} (f(u(x_2,t)) - f(u(x_1,t))) \, dt$$

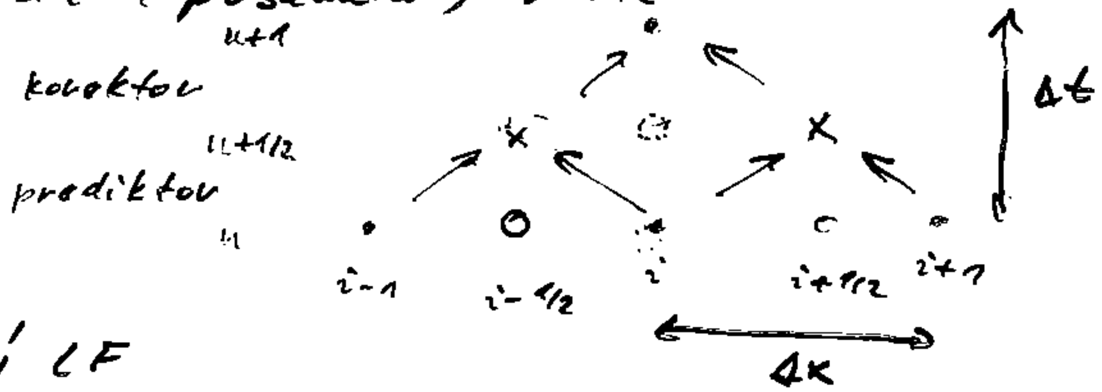
Diferenční schemata pro zákony zachování

$$u_t + f(u)_x = 0$$

• LF Lax-Friedrichs

$$\frac{u_i^{n+1} - \frac{u_{i+1}^n + u_{i-1}^n}{2}}{\Delta t} + \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2\Delta x} = 0$$

• poloviční síť (posunutá) o $\Delta x/2$



• dvoukrokový LF

prediktor

$$\frac{u_{i+1/2}^{n+1/2} - \frac{u_{i+1}^n + u_i^n}{2}}{\Delta t/2} + \frac{f(u_{i+1}^n) - f(u_i^n)}{\Delta x} = 0$$

korektor

$$\frac{u_i^{n+1} - \frac{u_{i+1/2}^{n+1/2} + u_{i-1/2}^{n+1/2}}{2}}{\Delta t/2} + \frac{f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})}{\Delta x} = 0$$

• dvoukrokový LW – Lax-Wendroff

prediktor – stejný jako LF prediktor

korektor

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})}{\Delta x} = 0$$

• LW pro advekční rovnici $f(u) = a \cdot u$

$$u_{i+1/2}^{n+1/2} = \frac{u_{i+1}^n + u_i^n}{2} - \frac{\Delta t}{2\Delta x} a \left(u_{i+1}^n - u_i^n\right)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2\Delta x}\left(u_{i+1}^n + \cancel{u_i^n} - \cancel{u_i^n} - u_{i-1}^n - \frac{\Delta t}{\Delta x} a \left(u_{i+1}^n - u_i^n - u_i^n + u_{i-1}^n \right.\right.$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{\Delta t}{2\Delta x^2} a^2 \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) = 0$$

LW pro adv. rovnici

- LF schema – 1 krokové $\quad u_t + f(u)_x = 0$

$$u_i^{n+1} = (u_{i+1}^n + u_{i-1}^n)/2 - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right)$$

$$u_i^{n+1} = u_i^n + \frac{\Delta x^2}{2}\underbrace{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}_{(u_{i+1}-u_i)-(u_i-u_{i-1})} - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right)$$

modif. rovnice $\quad u_i^{n+1} = u_i^n + F_{i+1/2}^n - F_{i-1/2}^n$

$$u_t + f(u)_x = \frac{\Delta x^2}{2\Delta t} u_{xx} \quad \to \text{difusní schema}$$

- LF 2-krokové schema

$$u_{i+1/2}^{n+1/2} = \frac{u_{i+1}^n + u_i^n}{2} - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1}^n) - f(u_i^n)\right)$$

$$u_i^{n+1} = \frac{u_{i+1/2}^{n+1/2} + u_{i-1/2}^{n+1/2}}{2} - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})\right)$$

$$u_i^{n+1} = \frac{u_{i+1}^n + 2u_i^n + u_{i-1}^n}{4} - \frac{\Delta t}{4\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})\right)$$

$$u_i^{n+1} = u_i^n + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\cdot\frac{\Delta x^2}{4} - \frac{\Delta t}{4\Delta x}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right) - \frac{\Delta t}{2\Delta x}\left(f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})\right)$$

modif. rovnice

$$u_t + f(u)_x = \frac{\Delta x^2}{4\Delta t} u_{xx} \quad \to \text{difusní schema}$$

– méně difusní než LF 1-krokové

- konzervativní tvar dif. schema je – konzervativní schema

$$u_i^{n+1} = u_i^n + (F_{i+1/2} - F_{i-1/2})$$

↑ numerický tok

- zachování veličin

$$\frac{d}{dt}\int_a^b u\,dx = F_b - F_a \qquad \sum u^{n+1} = \sum_{i=1}^{J} u_i^n + F_{J+1} - F_0$$

($F_{J+1} = F_b$, $F_0 = F_a$)

- pro LF-1 krokové

$$F_{i+1/2} = \frac{u_{i+1}^n - u_i^n}{2} - \frac{\Delta t}{2\Delta x} f(u_{i+1}^n)$$

- pro LF-2-krokové

$$F_{i+1/2} = \frac{u_{i+1}^n - u_i^n}{4} - \frac{\Delta t}{4\Delta x} f(u_{i+1}^n) - \frac{\Delta t}{2\Delta x} f(u_{i+1/2}^{n+1/2})$$

Burgersova rovnice

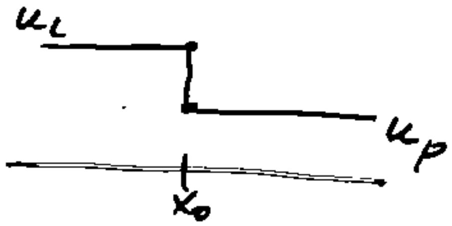
$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad\qquad u_t + f(u)_x = 0$$
$$f(u) = \frac{u^2}{2}$$
$$u_t + u \cdot u_x = 0$$

- charakteristika
$$x' = u \qquad\qquad x' = \frac{dx}{dt}$$
- na charakteristice je řešení konstantní
$$u(x,t) = u(x(t),t)$$

$$\frac{d}{dt} u(x(t),t) = u_t + x' \cdot u_x = u_t + u \cdot u_x = 0$$

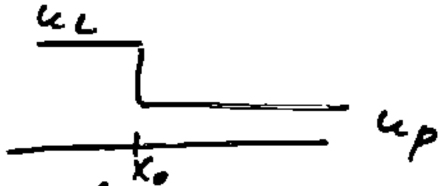
Riemannův problém

- počáteční podmínka
$$u(x,0) = \begin{cases} u_L & x < x_0 \\ u_p & x > x_0 \end{cases}$$

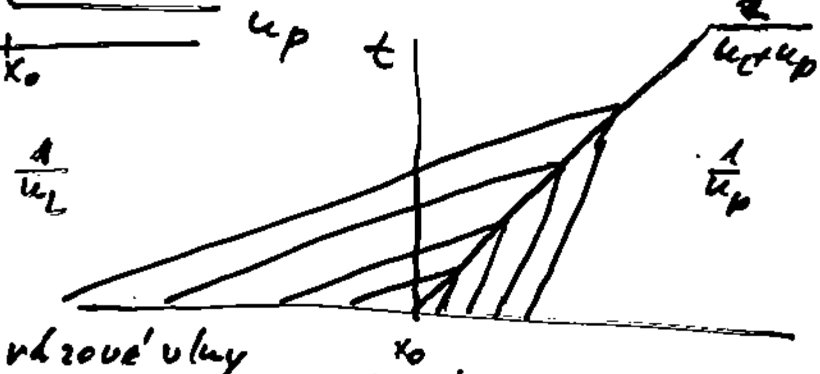


$$u_t + u \cdot u_x = 0$$

1. $u_L > u_p > 0$

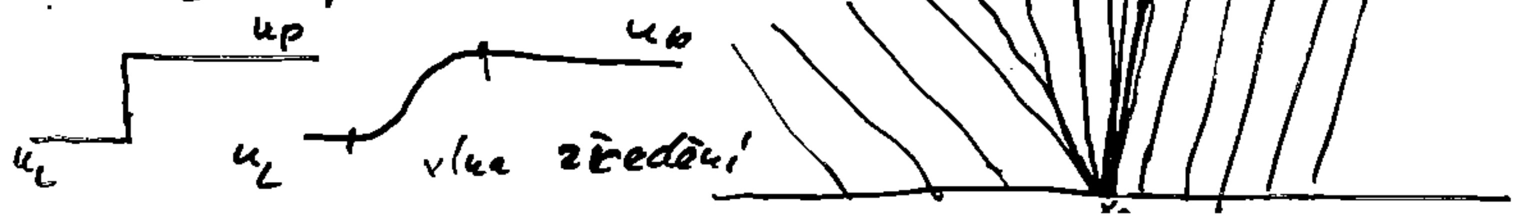


$$u(x,t) = \begin{cases} u_L, & x - x_0 < st \\ u_p, & x - x_0 > st \end{cases}$$



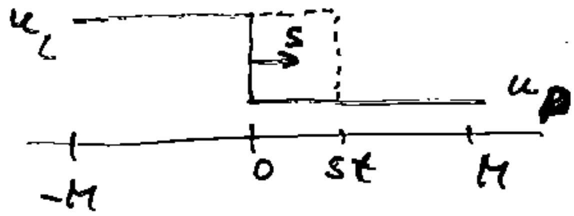
$$s = \frac{u_L + u_p}{2}$$ rychlost rázové vlny

2. $u_L < u_p$



vlna zředění

Rychlost rázové vlny



$$u_t + f(u)_x = 0$$

$$\int_{-M}^{M} u(x,t)_t \, dx + f(u_p) - f(u_L) = 0$$

$$\frac{\partial}{\partial t} \int_{-M}^{M} u(x,t) \, dx + f(u_p) - f(u_L) = 0$$

- vyjádříme $\int$

$$\int_{-M}^{M} u(x,t) \, dx = (M + st) u_L + (M - st) u_p$$

$$\frac{\partial}{\partial t} \int_{-M}^{M} u(x,t) \, dx = s (u_L - u_p)$$

- čili

$$s (u_L - u_p) + f(u_p) - f(u_L) = 0$$

$$\boxed{s = \frac{f(u_L) - f(u_p)}{u_L - u_p}}$$

Rankine - Hugoniotova podmínka

Burg. rovnice $f(u) = \frac{u^2}{2}$

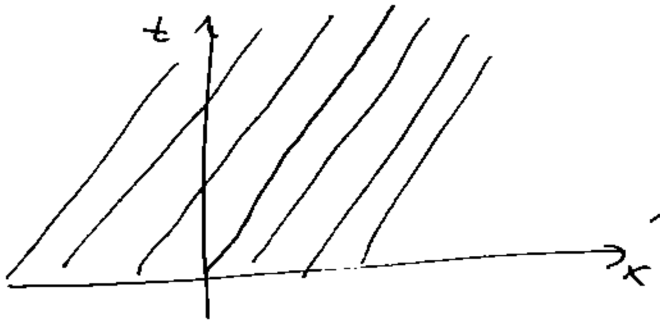
$$s = \frac{u_L^2 - u_p^2}{2(u_L - u_p)} = \frac{u_L + u_p}{2}$$

- pro systémy, skoky konzervativních veličin a toků na rázové vlně musí být lineárně závislé

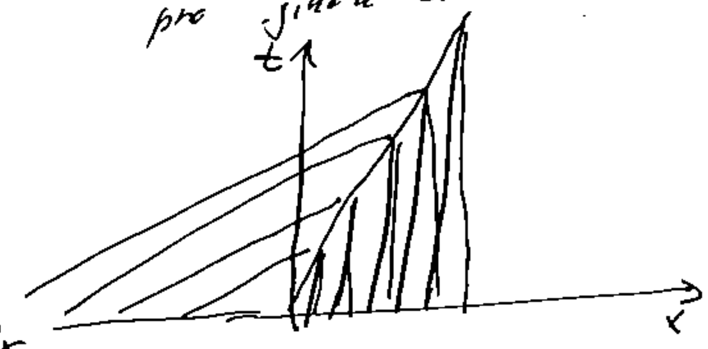
$$s = \frac{\vec{F}(\vec{U}_L) - \vec{F}(\vec{U}_p)}{\vec{U}_L - \vec{U}_p}$$

(3) kontaktní nespojitost
– pouze pro systémy

pro 1 složku



pro jinou složku



Systémy

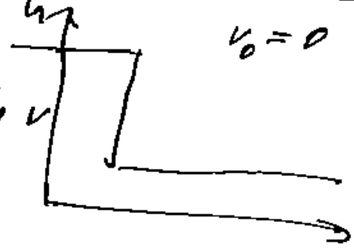
$$\vec{U}_t + \vec{F}(\vec{U})_x = 0 \quad \text{– konzervativní}$$

Rovnice mělké vody

$$\vec{U} = \begin{pmatrix} \varphi \\ m \end{pmatrix} \quad \vec{F}(\vec{u}) = \begin{pmatrix} m \\ \frac{m^2}{\varphi} + \frac{\varphi^2}{2} \end{pmatrix}$$

$\varphi = gh$

$m = ghv$



$$h_t + (hv)_x = 0$$

$$(hv)_t + (hv^2 + \tfrac{1}{2} g h^2)_x = 0$$

Eulerovy rovnice pro ideální plyn

$$\vec{u} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \quad \vec{F} = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}$$

stavová rovnice

$$E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho v^2$$

$$\gamma = 1.4 \,,\; 5/3$$

advekční tvar $\vec{u}_t + \frac{\partial \vec{F}}{\partial u} \cdot \vec{u}_x = 0$ ← Jacobián

pro mělkou vodu

$$\vec{F} = \begin{pmatrix} m \\ \frac{m^2}{\varphi} + \frac{\varphi^2}{2} \end{pmatrix} \quad \frac{\partial \vec{F}}{\partial u} = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\varphi^2} + \varphi & 2\frac{m}{\varphi} \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} \varphi \\ m \end{pmatrix}$$

$$\begin{pmatrix} \varphi \\ m \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\varphi^2} + \varphi & \frac{2m}{\varphi} \end{pmatrix} \begin{pmatrix} \varphi \\ m \end{pmatrix}_x = 0$$

$$A = \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\varphi^2}+\varphi, & \frac{2m}{\varphi} \end{pmatrix}$$

$$\det(\mathbb{J} - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -\frac{m^2}{\varphi^2}+\varphi, & \frac{2m}{\varphi} - \lambda \end{pmatrix} =$$

$$\varphi = gh$$
$$m = ghv$$

$$= \lambda^2 - \frac{2m}{\varphi}\lambda + \frac{m^2}{\varphi^2} - \varphi = 0$$

$$D = \frac{4m^2}{\varphi^2} - 4\frac{m^2}{\varphi^2} + 4\varphi = 4\varphi$$

$$\lambda_{1,2} = \frac{m}{\varphi} \pm \sqrt{\varphi} = v \pm \sqrt{gh}$$

vl. vektory

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t + \begin{pmatrix} \frac{m}{\varphi} + \sqrt{\varphi} & 0 \\ 0 & \frac{m}{\varphi} - \sqrt{\varphi} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_x = 0$$

$$w_{i_t} + \lambda_i^{(w)} w_{i_x} = 0$$

$$w_{i_t} + a_i(w)\, w_{i_x} = 0$$

Rovnice mělké vody $\quad \vec{u}_t + \vec{f}(\vec{u})_x = 0$

$$\begin{pmatrix} \varphi \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ \frac{m^2}{\varphi} + \frac{\varphi^2}{2} \end{pmatrix}_x = 0 \qquad \begin{matrix} \varphi = gh \\ m = ghv \end{matrix}$$

- Rankin–Hugoniotova podmínka

$$s = \frac{\vec{f}(\vec{u}_l) - \vec{f}(\vec{u}_p)}{\vec{u}_l - \vec{u}_p}$$

- po složkách

$$s = \frac{m_l - m_p}{\varphi_l - \varphi_p} = \frac{\frac{m_l^2}{\varphi_l} + \frac{\varphi_l^2}{2} - \frac{m_p^2}{\varphi_p} - \frac{\varphi_p^2}{2}}{m_l - m_p}$$

- zvolíme

$$\varphi_l = 4, \quad \varphi_p = 2, \quad m_p = 0$$

$$\frac{m_l}{2} = \frac{\frac{m_l^2}{4} + 8 - 2}{m_l}$$

$$\frac{m_l^2}{2} = \frac{m_l^2}{4} + 6$$

$$2m_l^2 = m_l^2 + 24$$

$$m_l^2 = 24$$

$$m_l = \pm 2\sqrt{6}$$

→ Riemannův problém jehož řešením je jedna rázová vlna s rychlostí

$$s = \pm\sqrt{6}$$

Eulerovy rovnice $\vec{u}_t + F(u)_x = 0$

$$\vec{u} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix} \quad \vec{F} = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(E+p) \end{pmatrix} = \begin{pmatrix} m \\ \frac{m^2}{\rho}\left(1 - \frac{\gamma-1}{2}\right) + (\gamma-1)E \\ \frac{m}{\rho}\left(\gamma E - \frac{\gamma-1}{2}\frac{m^2}{\rho}\right) \end{pmatrix}$$

$$p = (\gamma-1)\left(E - \tfrac{1}{2}\rho v^2\right) = (\gamma-1)\left(E - \frac{m^2}{2\rho}\right) \quad m = \rho u$$

$$E = \rho\varepsilon + \tfrac{1}{2}\rho u^2$$

$$p = (\gamma-1)\rho\varepsilon$$

$$\gamma = 1.4,\ 5/3$$

$$\vec{F} = \begin{pmatrix} m \\ \frac{m^2}{2\rho}(3-\gamma) + (\gamma-1)E \\ \frac{m}{\rho}\left(\gamma E - \frac{(\gamma-1)m^2}{2\rho}\right) \end{pmatrix}$$

$$J = \frac{\partial\vec{F}}{\partial\vec{u}} = \begin{pmatrix} 0, & 1, & 0 \\ -\frac{m^2}{2\rho^2}(3-\gamma), & \frac{m}{\rho}(3-\gamma), & \gamma-1 \\ \frac{(\gamma-1)m^3}{\rho^3} - \frac{\gamma m E}{\rho^2} & -\frac{3(\gamma-1)m^2}{2\rho^2} + \frac{\gamma E}{\rho} & \frac{\gamma m}{\rho} \end{pmatrix}$$

$$\det(J - \lambda I) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ -\frac{m^2}{2\rho^2}(3-\gamma), & \frac{m}{\rho}(3-\gamma) - \lambda, & \gamma-1 \\ \frac{(\gamma-1)m^3}{\rho^3} - \frac{\gamma m E}{\rho^2}, & -\frac{3(\gamma-1)m^2}{2\rho^2} + \frac{\gamma E}{\rho}, & \frac{\gamma m}{\rho} - \lambda \end{pmatrix}$$

$$= -\lambda\left(v(3-\gamma) - \lambda\right)\left(\gamma v - \lambda\right) + (\gamma-1)\left((\gamma-1)v^3 - \gamma v\frac{E}{\rho}\right) +$$
$$\lambda(\gamma-1)\left(\gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)v^2\right) + \frac{1}{2}v^2(3-\gamma)(\gamma v - \lambda) =$$

$$= -\lambda^3 + \lambda^2\left(v(3-\gamma) + \gamma v\right) - \lambda v^2\gamma(3-\gamma)$$
$$+ (\gamma-1)^2v^3 - \gamma(\gamma-1)v\frac{E}{\rho} + \lambda(\gamma-1)\left(\gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)v^2\right)$$
$$+ \frac{1}{2}v^3\gamma(3-\gamma) - \frac{1}{2}\lambda v^2(3-\gamma) =$$

$$= -\lambda^3 + 3v\lambda^2 + \lambda\left[-v^2\gamma(3-\gamma) + (\gamma-1)\gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)^2v^2\right.$$
$$\left. - \frac{1}{2}v^2(3-\gamma)\right] + \frac{v^3}{2}\left(2\gamma^2 - 4\gamma + 2 + 3\gamma - \gamma^2\right)$$
$$- \gamma(\gamma-1)v\frac{E}{\rho} =$$

$$\det = -\lambda^3 + 3v\lambda^2 + \lambda\left[\frac{v^2}{2}\left(2\gamma^2 - 6\gamma - 3\gamma^2 + 6\gamma - 3 - 3 + \gamma\right) + (\gamma-1)\gamma\frac{E}{\rho}\right]$$

$$+\frac{v^3}{2}\left(\gamma^2 - \gamma + 2\right) - \gamma(\gamma-1)v\frac{E}{\rho} =$$

$$= -\lambda^3 + 3v\lambda^2 + \lambda\left[\frac{v^2}{2}\left(-\gamma^2 + \gamma - 6\right) + (\gamma-1)\gamma\frac{E}{\rho}\right]$$

$$+\frac{v^3}{2}\left(\gamma^2 - \gamma + 2\right) - \gamma(\gamma-1)v\frac{E}{\rho} = 0$$

$\det(\lambda = v) = 0 \qquad \lambda_1 = v$

$$\frac{\det}{\lambda - v} = -\lambda^2 + 2v\lambda - v^2 + \gamma^2\frac{E}{\rho} - \frac{1}{2}v^2\gamma^2 - \gamma\frac{E}{\rho} + \frac{1}{2}v^2\gamma$$

$$= -\lambda^2 + 2v\lambda - v^2 + \gamma(\gamma-1)\left(\frac{E}{\rho} - \frac{1}{2}v^2\right)$$

$p = (\gamma-1)\left(E - \frac{1}{2}\rho v^2\right)$

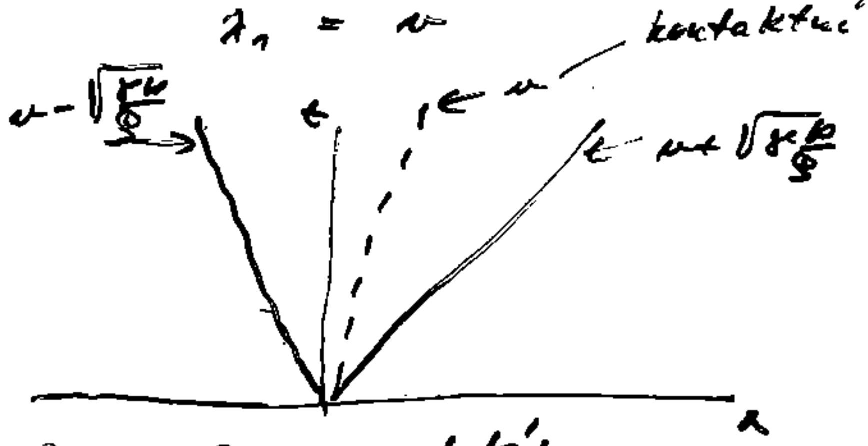
$$= -\lambda^2 + 2v\lambda - v^2 + \gamma\frac{p}{\rho}$$

$$D = 4v^2 - 4v^2 + 4\gamma\frac{p}{\rho} = 4\gamma\frac{p}{\rho}$$

$$\lambda_{2,3} = \frac{-2v \pm 2\sqrt{\gamma\frac{p}{\rho}}}{-2} = v \mp \sqrt{\gamma\frac{p}{\rho}}$$

$\lambda_1 = v$

$a_s = \sqrt{\frac{\gamma p}{\rho}}$ rychlost zvuku



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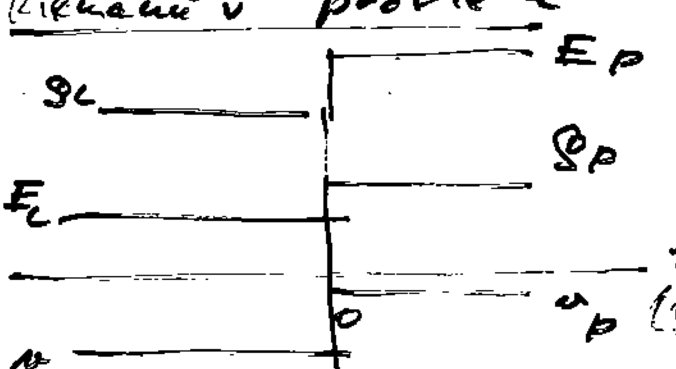
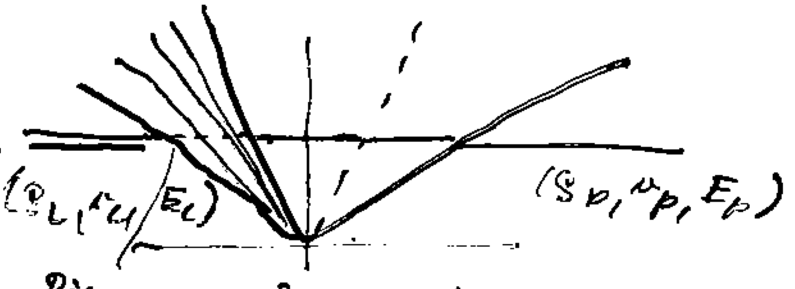
Riemannův problém



Lze analyticky dopočítat řešení lib. RP



Riemannovy invarianty

3 1D tests

3.1 Description of 1D problems

For 1D tests we have chosen five 1D (in x) Riemann problems from [10], tests *1,2,4,5,6* plus four others: test *1-tvj* is the same as test *1* but with a large jump in the y-velocity; Noh is the classical 1D Noh problem [38]; test *3a* is a modification of test 3 from [10] keeping a stationary contact; *peak* is a hard problem with strong narrow peak in density found by Milan Kuchařík [39]), and the Woodward-Collela blast wave problem [35]. All the 1D problems except the blast wave problem are simple Riemann problems with known exact solutions.

All codes are 2D-capable, that is, they have two velocity components. The Riemann problems are on the interval $x \in (0,1)$ (except for peak which is computed on $x \in (0.1, 0.6)$) with initial discontinuity at $x_0 \in (0,1)$ solved for time $t \in (0,T)$. The initial conditions are given by constant left state (ρ_L, u_L, v_L, p_L) of density, x-velocity, y-velocity, and pressure on the interval $x \in (0, x_0)$ and right state (ρ_R, u_R, v_R, p_R) on the interval $x \in (x_0, 1)$. Each test is defined by the ten parameters $\rho_L, u_L, v_l, p_L, \rho_R, u_R, v_R, p_R, x_0, T$. For all 1D Riemann problems except test *1-tvj*, the data are given in Table 1, together with $v_L = v_R$=0. The data for test *1-tvj* is the same as for test test *1*, together with $v_L = 1, v_R = -5$. The Noh problem uses the gas constant $\gamma = 5/3$ while all other tests use $\gamma = 1.4$. All Riemann problem tests use natural boundary conditions.

Test	ρ_L	u_L	p_L	ρ_R	u_R	p_R	x_0	T
1	1	0.75	1	0.125	0	0.1	0.3	0.2
2	1	-2	0.4	1	2	0.4	0.5	0.15
Noh	1	1	10^{-6}	1	-1	10^{-6}	0.5	1
3a	1	-19.59745	1000	1	-19.59745	0.01	0.8	0.012
4	5.99924	19.5975	460.894	5.99242	-6.19633	46.095	0.4	0.035
5	1.4	0	1	1	0	1	0.5	2
6	1.4	0.1	1	1	0.1	1	0.5	2
peak	0.1261192	8.9047029	782.92899	6.591493	2.2654207	3.1544874	0.5	0.0039

Table 1: Definition of 1D Riemann problem tests

The classic Woodward-Collela blast wave problem [35] computes the interaction of waves from two Riemann problems with reflecting boundary conditions. The problem is treated again on the interval $x \in (0,1)$. Two initial discontinuities are located at $x_1 = 0.1$ and $x_2 = 0.9$. The initial density is one and the velocity is zero everywhere. Initial pressures in three different regions (left p_l, middle p_m and right p_r) are $(p_l, p_m, p_r) = (1000, 0.01, 100)$.

For the numerical treatment of most test problem we use 100 grid cells, exceptions being tests *3a* and *4* using 200 cells, *blast* using 400 and 2000 cells and *peak* using 800 cells.

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Figs. 6.8 to 6.11 show comparisons between exact solutions (line) and numerical solutions (symbol) at a given output time obtained by the Godunov method, for all four test problems. The quantities shown are density ρ, particle speed u, pressure p and specific internal energy e. For comparison, we also solved these test problems using the Lax–Friedrichs method, see Figs. 6.12 to 6.15, and the the Richtmyer method.This latter method failed to provide a solution to Tests 2 to 4. For Test 1 the solution is shown in Fig. 6.16.

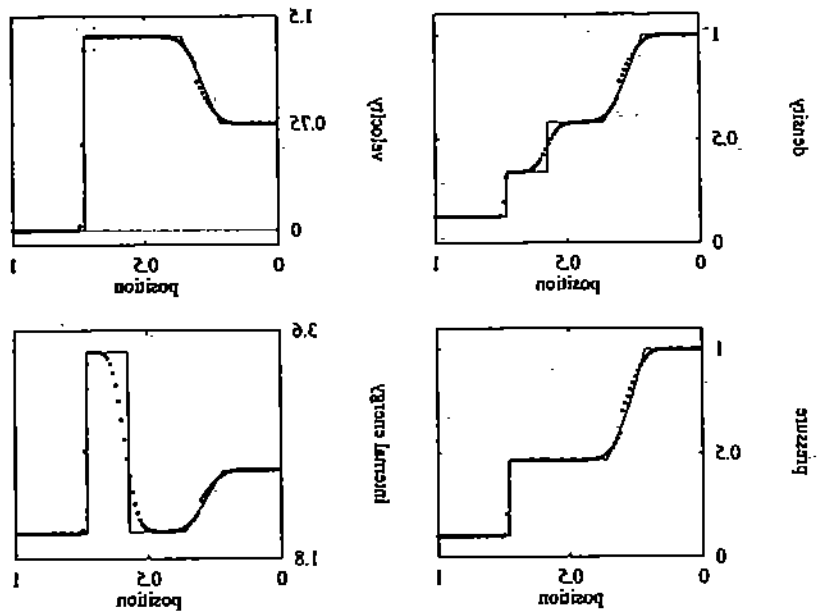


Fig. 6.8. Godunov's method applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at the output time 0.2 units

6.4.1 Numerical Results for Godunov's Method

The results for Test 1, shown in Fig. 6.8, are typical of the Godunov's first–order accurate method described in this chapter.

The numerical approximation of the shock wave, of zero–width transition in the exact solution, has a transition region of width approximately $4\Delta x$; that is, the shock has been smeared over 4 computing cells. This spreading of shock waves may seem unsatisfactory, but it is quite typical of numerical solutions; in fact most first–order methods will spread a shock wave even more. A satisfactory feature of the numerical shock wave of Fig. 6.8 is that it is monotone, there are no spurious oscillations in the vicinity of the shock, at least

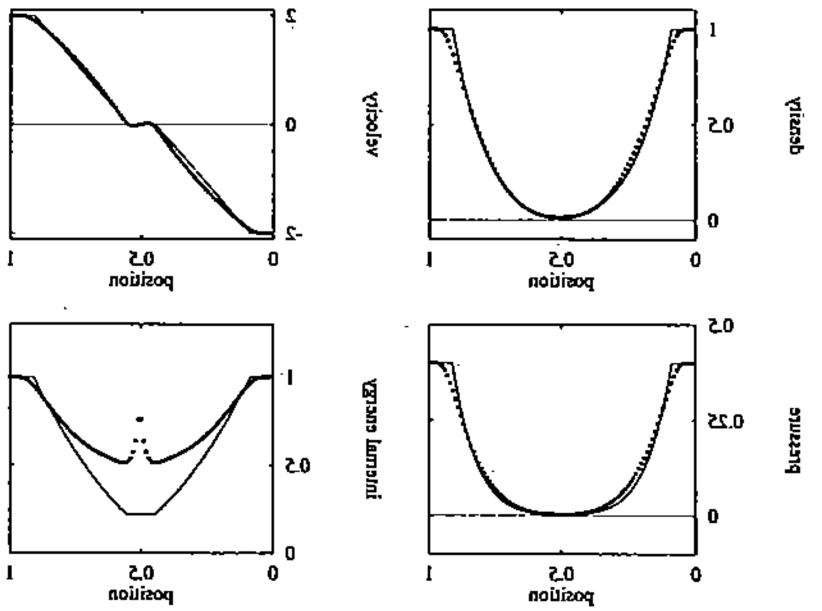


Fig. 6.9. Godunov's method applied to Test 2, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared at the output time 0.15 units

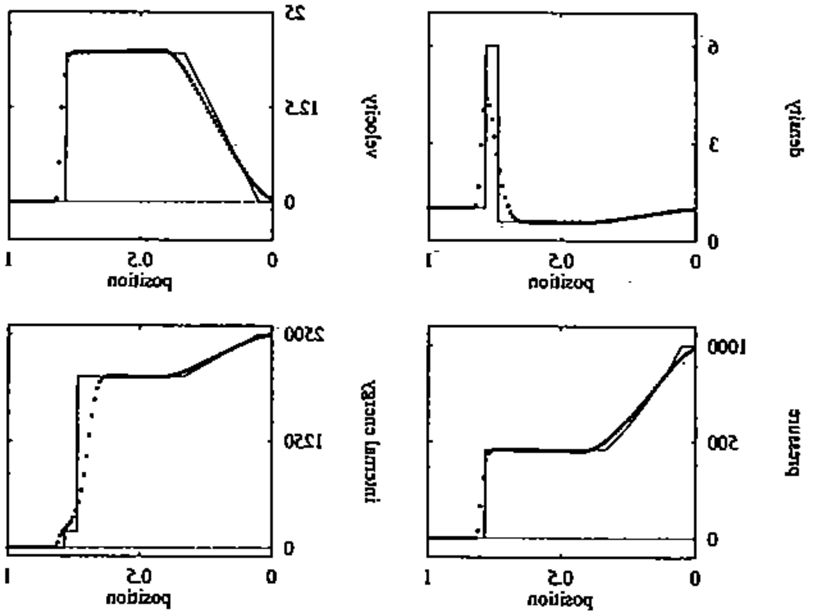


Fig. 6.10. Godunov's method applied to Test 3, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared ...

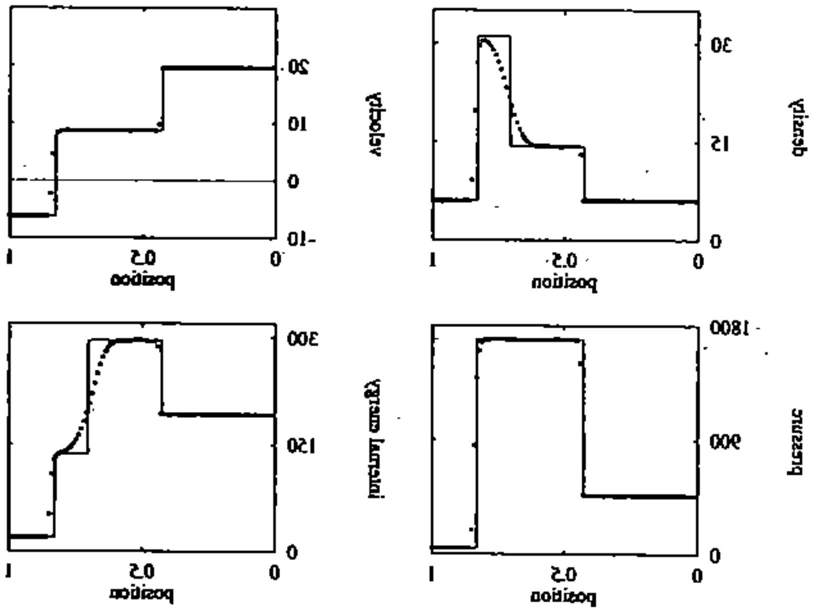


Fig. 6.11. Godunov's method applied to Test 4, with $x_0 = 0.4$. Numerical (symbol) and exact (line) solutions are compared at the output time 0.035 units

for this example. Monotonicity of shock waves computed by the Godunov method depends on the speed of the shock and it holds in most cases except when the shock speed is very close to zero. The contact discontinuity, seen in the density and internal energy plots, is smeared over 20 cells; generally contact waves are much more difficult to resolve accurately than shock waves. This is due to the linear character of contacts; characteristics on either side of the wave run parallel to the wave. In shock waves characteristics on either side of the wave run into the shock, a compression mechanism that helps the numerical resolution of shock waves. As for the shock case, the solution for the contact is perfectly monotone.

Another positive feature of the numerical approximation of the discontinuities is that their speed of propagation is correct and thus their average positions are correct. This is a consequence of the conservative character of Godunov's method. The rarefaction wave is a smooth flow feature and is reasonably well approximated by the method except near the head and the tail, where a discontinuity in derivative exists. Another visible error in the rarefaction is the small discontinuous jump within the rarefaction. This is sometimes referred to as the *entropy glitch* and arises only in the presence of sonic rarefaction waves, as in the present case. Godunov's method is theoretically entropy satisfying [148] and we therefore expect the size of the jump in the entropy glitch to tend to zero as the mesh size Δx tends to zero.

are on the interval $x \in (0,1)$ (except
discontinuity at $x_0 \in (0,1)$ solved for
left state (ρ_L, u_L, v_L, p_L) of density, x
and right state (ρ_R, u_R, v_R, p_R) on the
eters $\rho_L, u_L, v_l, p_L, \rho_R, u_R, v_R, p_R, x_0, T$
are given in Table 1, together with v_L
1, together with $v_L = 1, v_R = -5$. Th
tests use $\gamma = 1.4$. All Riemann proble

Test	ρ_L	u_L	
1	1	0.75	
2	1	-2	
Noh	1	1	
3a	1	-19.59745	
4	5.99924	19.5975	46
5	1.4	0	
6	1.4	0.1	
peak	0.1261192	8.9047029	782

Table 1: Defi

The classic Woodward-Collela blas
two Riemann problems with reflecting
interval $x \in (0,1)$. Two initial discor
density is one and the velocity is zero
p_l, middle p_m and right p_r) are $(p_l, p_m$
For the numerical treatment of mc
3a and *4* using 200 cells, *blast* using 4

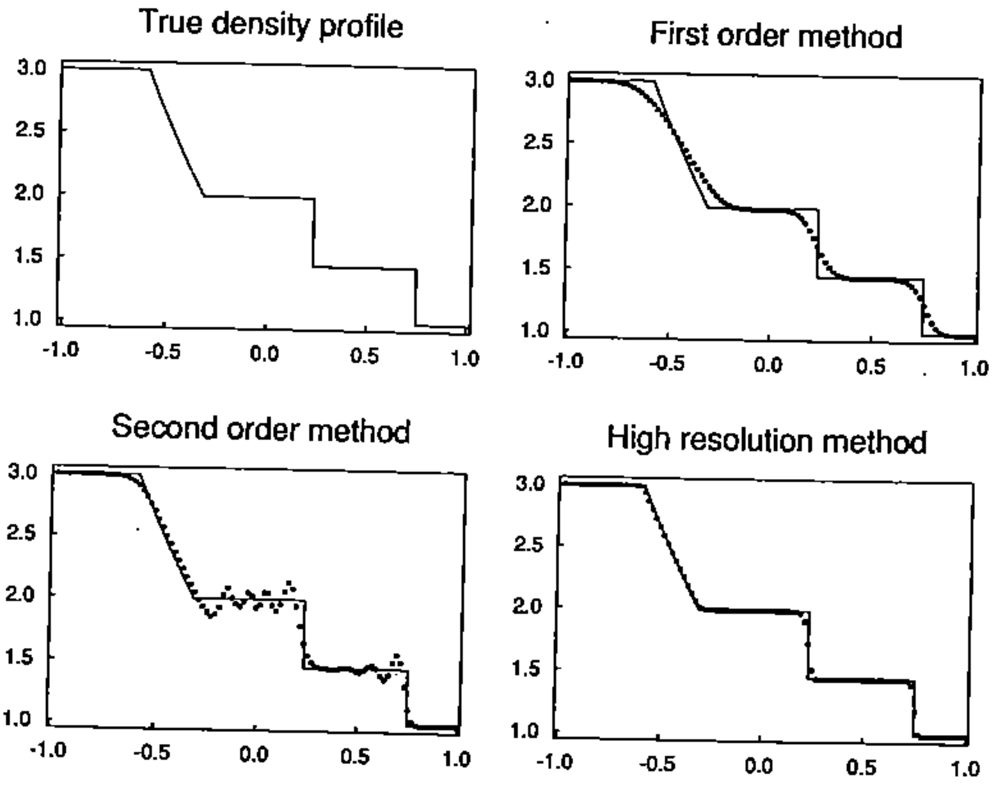


Figure 1.4. Solution of the shock tube problem at $t = 0.5$. The true density profile is shown along with the numerical solution computed with three different methods: Godunov's method (first order), MacCormack's method (second order), and a high resolution method.

Nessyahu – Tadmor – NT schema

$$u_t + f(u)_x = 0$$

$$u_j^{n+1/4} = u_j^n - \frac{\Delta t}{4\Delta x} f_j^{\prime}$$

$$u_{j+1/2}^{n+1/2} = \frac{u_j^n + u_{j+1}^n}{2} + \frac{u_j^{n\prime} - u_{j+1}^{n\prime}}{8} - \frac{\Delta t}{2\Delta x}\left(f(u_{j+1}^{n+1/4}) - f(u_j^{n+1/4})\right)$$

$$u_{j+1/2}^{n+3/4} = \dots \text{ posunuté}$$

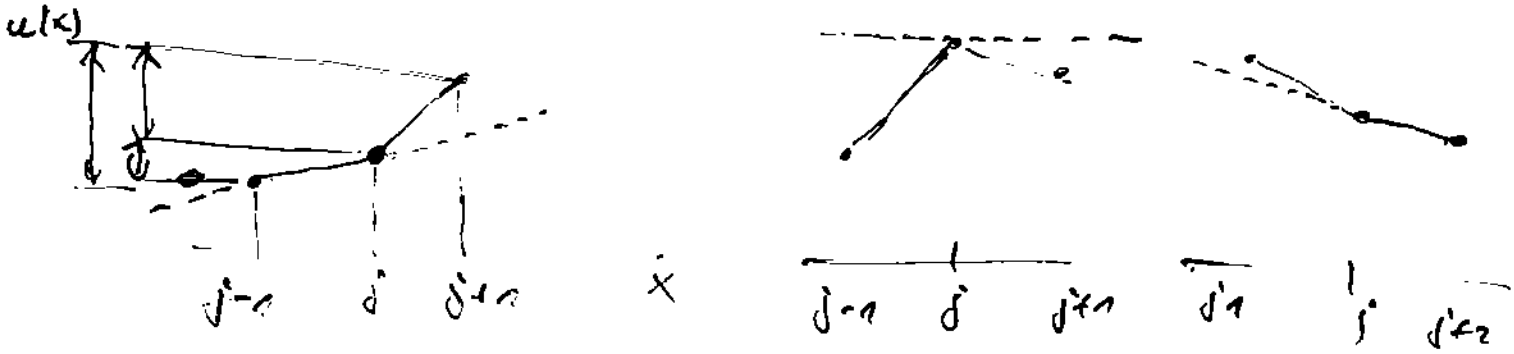
$$u_j^{n+1} = \dots \quad -\,''\,-$$

min mod limiter, $\theta \in (1,2)$

$$u_j^{\prime} = MM\left(\theta(u_{j+1} - u_j),\ \frac{u_{j+1} - u_{j-1}}{2},\ \theta(u_j - u_{j-1})\right)$$

$$MM(x_1, x_2, x_3) = \begin{cases} \min x_j, & x_j > 0 \ \forall j \\ \max x_j, & x_j < 0 \ \forall j \\ 0 & \text{jinde} \end{cases}$$

$\theta = 1$



1. $f_j^{\prime} = A_j^n u_j^{\prime}$ $\qquad A_j = \frac{\partial f}{\partial u}$
2. limiter přímo na složky f, $f_j^{\prime} = MM(\theta(f_{j+1} - f_j)$ ...