Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

No constraints are assumed – $q_i = x_i$ and $\dot{q}_i$ = v_i

Lagrange function ***L = T – U*** T – kinetic energy, V – potential energy

Electric field is potential $$L = \frac{mv^2}{2} - q\phi$$

Magnetic field is not potential

Instead of $$\vec{E}, \vec{B} - \longrightarrow \phi, \vec{A}$$

$$\vec{B} = \nabla \times \vec{A} = curl\,\vec{A} \longrightarrow div\,\vec{B} = 0$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \rightarrow \quad \nabla \times \vec{E} = -\frac{\partial}{\partial t}\nabla \times \vec{A} = -\frac{\partial \vec{B}}{\partial t}$$

Symbol “rot” means “curl”

gauge invariance

$$\vec{A}' = \vec{A} + \nabla\psi$$

$$\phi' = \phi - \frac{\partial \psi}{\partial t}$$

Lorenz gauge (Ludwig Lorenz)	Coulomb gauge
$div\,\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$	$div\,\vec{A} = 0$
$\Delta\phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$	$\Delta\phi = -\rho/\epsilon_0$
$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{j}$	$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{j} + \frac{1}{c^2}\nabla\frac{\partial \phi}{\partial t}$

We introduce **4-vectors**

$$A^{\mu} = \begin{pmatrix} \phi/c \\ \vec{A} \end{pmatrix} \qquad J^{\mu} = \begin{pmatrix} c\rho \\ j \end{pmatrix} = \begin{pmatrix} qc\delta(\vec{x}-\vec{x}_0) \\ q\vec{v}\delta(\vec{x}-\vec{x}_0) \end{pmatrix}$$

Lagrangian density

$$\mathcal{L} = J^{\mu}A_{\mu} = -q\phi\delta(\vec{x}-\vec{x}_0) + q(\vec{A}\vec{v})\delta(\vec{x}-\vec{x}_0)$$

$$\tilde{L} = \int \mathcal{L}\, d^3x = -q\phi + q\vec{A}\vec{v}$$

$$L = \frac{mv^2}{2} - q\phi + q\vec{A}\vec{v}$$

$$\frac{\partial L}{\partial v_i} = mv_i + qA_i \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial v_i} = m\dot{v}_i + q\dot{A}_i + q(\vec{v}\nabla)A_i$$

$$\frac{\partial L}{\partial x_i} = -q\frac{\partial \phi}{\partial x_i} + qv_j\frac{\partial A_j}{\partial x_i}$$

$\Rightarrow$ Equation of motion

$$m\dot{v}_i = -q\underbrace{\left(\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i}\right)}_{-E_i} \underbrace{- qv_j\frac{\partial A_i}{\partial x_j} + qv_j\frac{\partial A_j}{\partial x_i}}_{q(v\times B)_i}$$

Derivation of magnetic force (x component)

$$F_x^{(B)} = \cancel{qv_x\frac{\partial A_x}{\partial x}} - qv_y\frac{\partial A_y}{\partial x} + qv_z\frac{\partial A_z}{\partial x} - \cancel{qv_x\frac{\partial A_x}{\partial x}} -$$
$$- qv_y\frac{\partial A_x}{\partial y} - qv_z\frac{\partial A_x}{\partial z} = qv_y\underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{B_z} -$$
$$- qv_z\underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{B_y} = q(\vec{v}\times\vec{B})_x$$

Particle momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + q\vec{A}$$

$$\mathcal{E} = \vec{v}\frac{\partial L}{\partial \vec{v}} - L = mv^2 + q\vec{A}\vec{v} - \frac{1}{2}mv^2 + q\phi - q\vec{A}\vec{v}$$

$$= \frac{mv^2}{2} + q\phi = \underbrace{\frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi}_{H} = c$$

$$\vec{\dot{x}} = \frac{\partial H}{\partial \vec{p}} \qquad \vec{\dot{p}} = -\frac{\partial H}{\partial \vec{x}}$$

$$\dot{x} = \frac{\vec{p} - q\vec{A}}{m} = V$$

$$\vec{\dot{p}} = +\frac{(\vec{p} - q\vec{A})}{m} q \frac{\partial \vec{A}}{\partial \vec{x}} - q\frac{\partial \phi}{\partial \vec{x}}$$

$$\frac{dp_i}{dt} = q\underbrace{\frac{(p_j - qA_j)}{m}}_{v_j}\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

$$m\frac{dv_i}{dt} + \underbrace{q\frac{dA_i}{dt}}_{q\frac{\partial A_i}{\partial t} + qv_j\frac{\partial}{\partial x_j}A_i} = qv_j\frac{\partial A_j}{\partial x_i} - q\frac{\partial \phi}{\partial x_i}$$

$$\Rightarrow \quad m\frac{dv_i}{dt} = q\vec{E} + q(\vec{v} \times \vec{B})$$

Relativistic description

Field part – relativistic invariant – OK, kinetic part – modified to be invariant

$$S = \int L\, dt \quad \text{– invariant}$$

$$\Rightarrow L_p\, dt = \alpha \sqrt{ds^2} = \alpha \sqrt{1 - v^2/c^2}\, dt$$

Constant α can be found from classic v << *c* limit

$$L_p = \alpha \sqrt{1 - \frac{v^2}{c^2}} \simeq \alpha \left(1 - \frac{v^2}{2c^2}\right) = \alpha - \alpha \frac{v^2}{2c^2}$$

$$-\alpha \frac{v^2}{2c^2} = \frac{m v^2}{2} \Rightarrow \alpha = -m c^2$$

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{A}\vec{v}$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + q\vec{A}$$

$$\mathcal{E} = \vec{v} \frac{\partial L}{\partial \vec{v}} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + q\phi$$

$$\frac{1}{c^2}(\mathcal{E} - q\phi)^2 - (p - q\vec{A})^2 = \frac{m(c^2 - v^2)}{1 - \frac{v^2}{c^2}}$$

$$\mathcal{E} = c\sqrt{m^2c^2 + (\vec{p} - q\vec{A})^2} + q\phi$$