The renormalization group algorithm in problems of laser-plasma interactions

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Abstract

- Evolution of the renormgroup algorithm and the related renormgroup symmetry, introduced in mathematical physics for solutions of boundary-value problems based on differential equations, is reviewed. We discuss the essential progress made recently in the application of this algorithm to models with integral equations.
- Several physical illustrations from nonlinear optics and plasma physics demonstrate the potentialities of the algorithm for models based on differential equations and models with nonlocal terms in the form of the linear solution functionals.

Outline

- Introduction to Renormalization Group (RG)
  Symmetries: history, foundations, evolution of concept, etc.

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- Generalization of the RG algorithm to nonlocal models.
- Illustrations of the RG algorithm: applications to various problems in mathematical physics.

General grasp of RGS

Wave beam in nonlinear medium

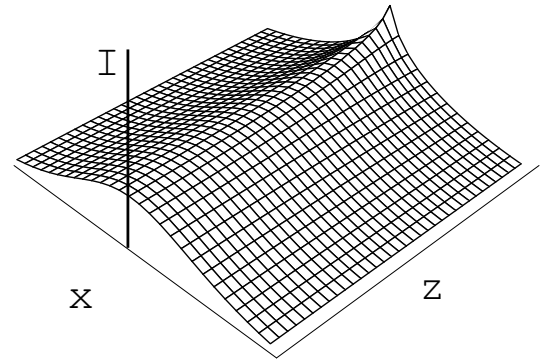
$$k_z + kk_x = \alpha I_x \,,\; I_z + (kI)_x = 0 \,,$$

$$k(0,x) = 0 \,, \quad I(0,x) = I_0(x).$$

$E = A\exp(ik_0\Psi)$ – electric field;

$k \sim \Psi_x$ – eikonal gradient;

$I \sim |\, A \,|^2$ – wave beam intensity.



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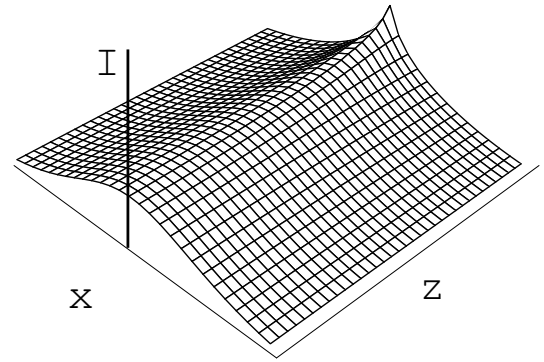

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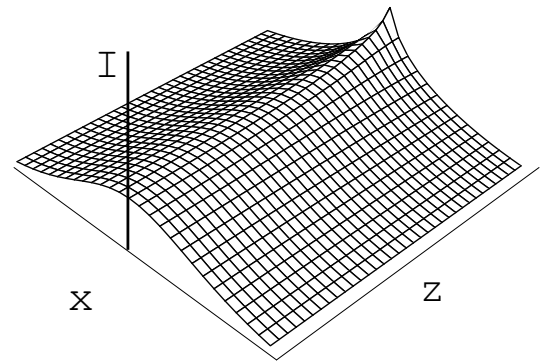

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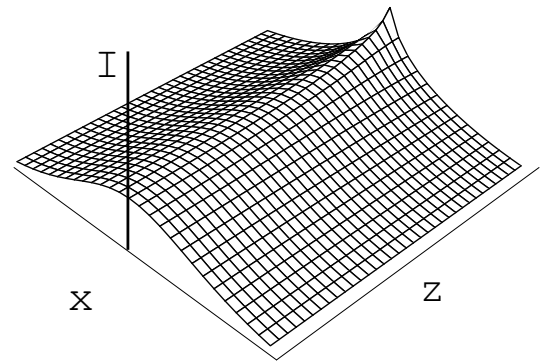

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$$\mathcal{I}(z) = \int \delta(x) I(z,x)\mathrm{d}x\,.$$

RGM and RGS: historical introduction

’51-’55

Renormalization
in QFT

RGM and RGS: historical introduction

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$\Downarrow$

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Phys.Rep., v.352 (2001), 219

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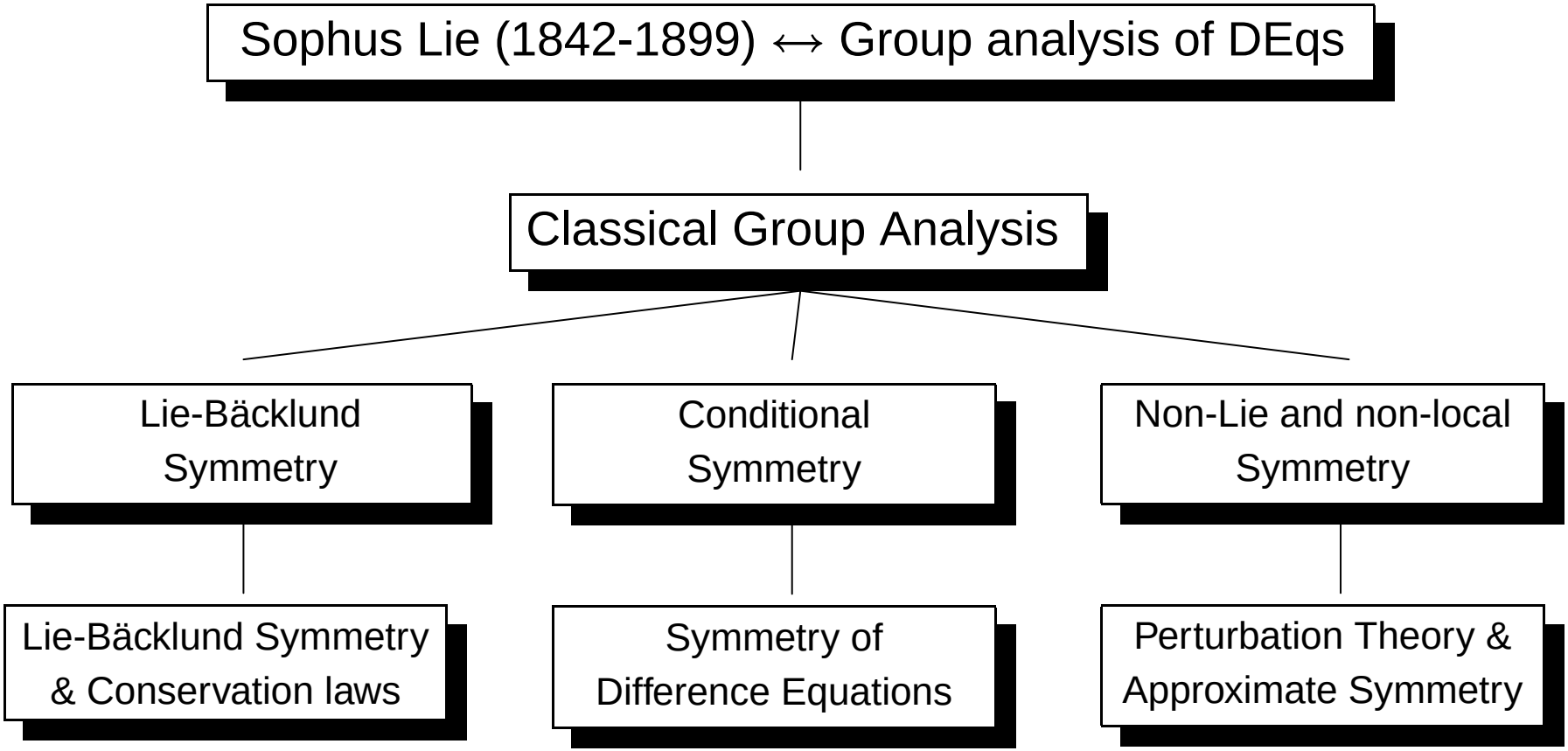
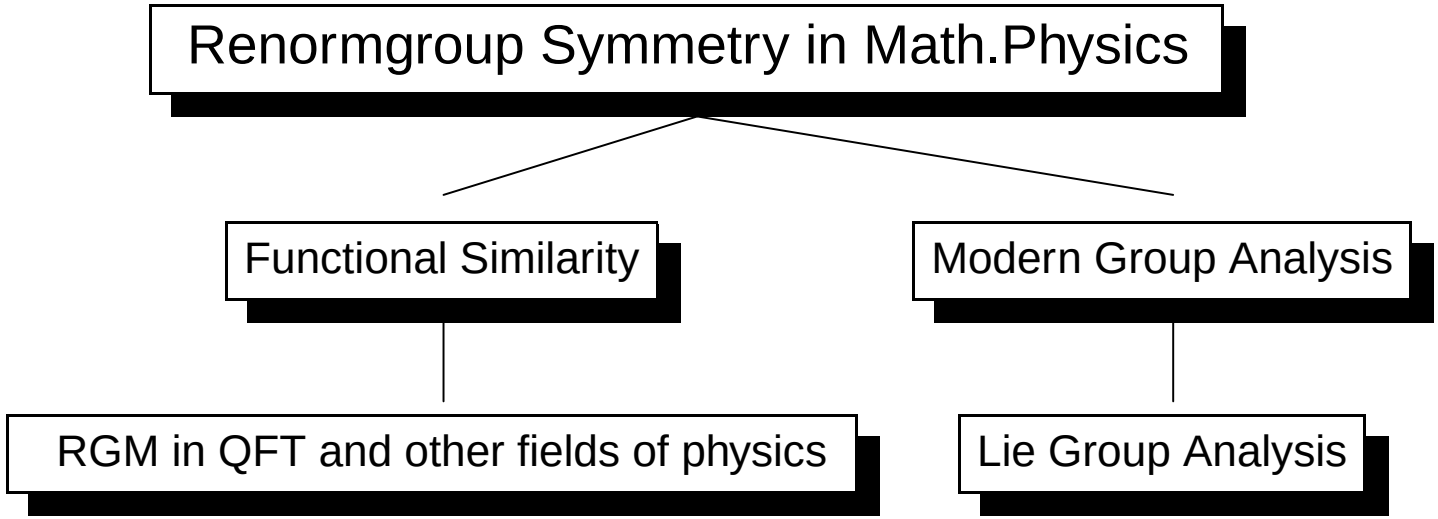
From S.Lie to Modern Group Analysis



Evolution of the RGS concept



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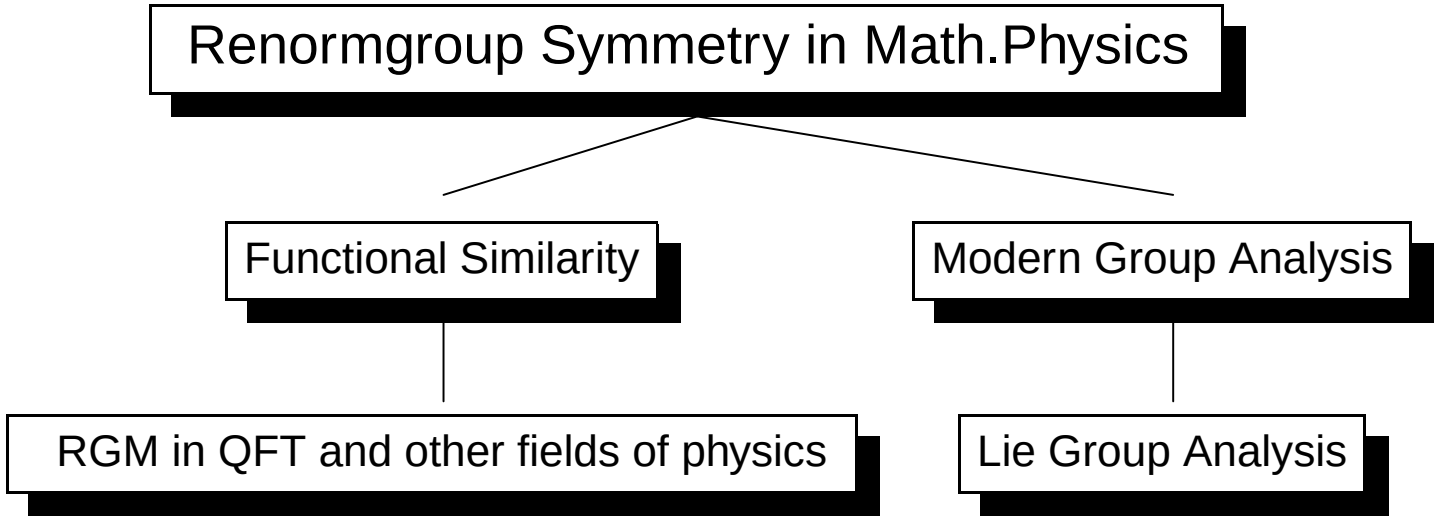

The characteristic features of RGS:

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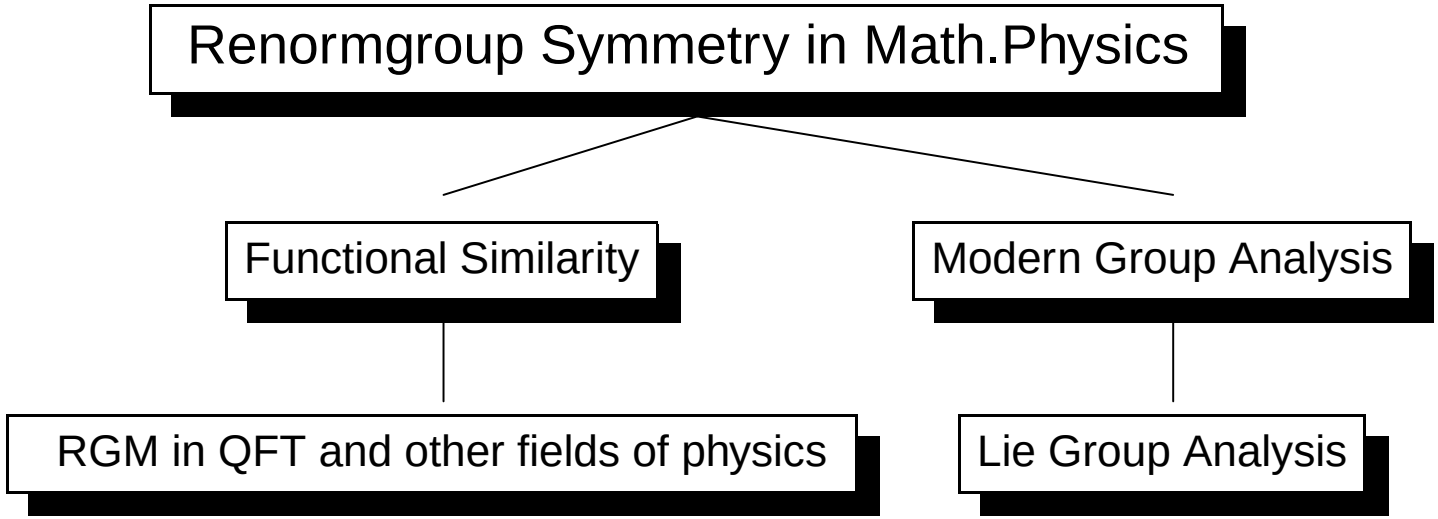

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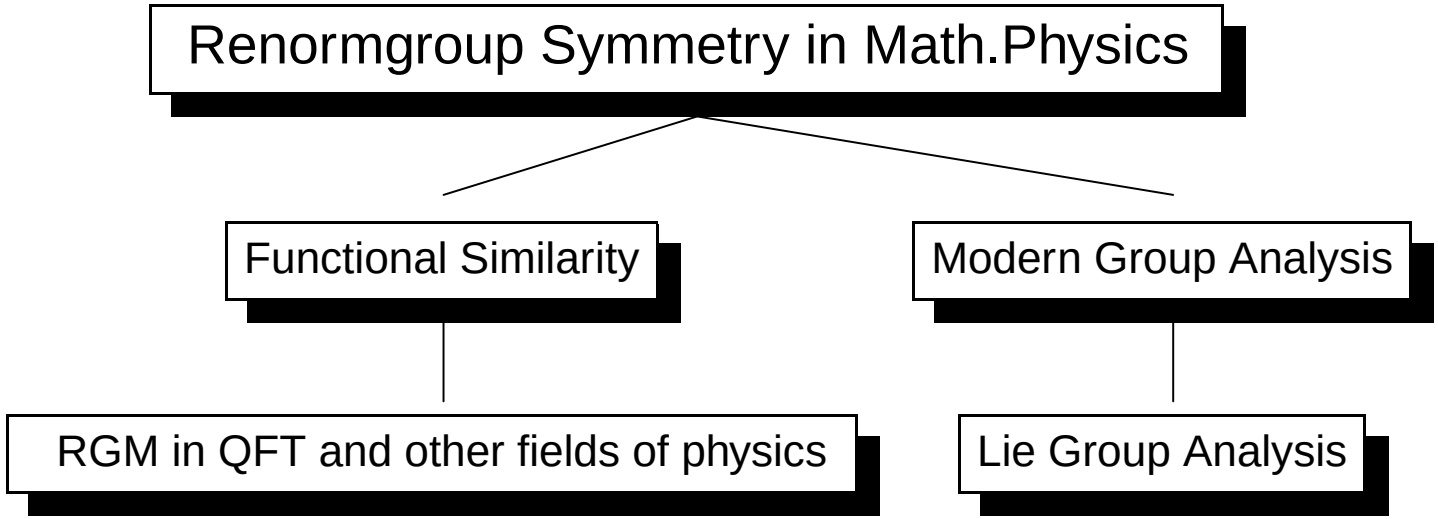

The characteristic features of RGS:

- It extends the notion of RG and generalizes the form of RG implementation that differ from that known, say, in Quantum Field Theory.
- It is obtained with the use of a regular algorithm based on the methods of Modern Group Analysis (MGA).

Kovalev,Pustovalov,Shirkov, JMP,98; Kovalev,Shirkov,Phys.Rep.02

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The FS transformation in QFT is the scaling transformation of an independent variable accompanied by a functional transformation of the solution characteristic. It is introduced by means of either finite transformations

$$R_t \quad : \quad \{ \, x \to x' = x/t \; , \quad g \to g' = \bar{g}(t, g) \, \} \; ,$$

or the infinitesimal operator

$$X = x\partial_x - \beta(g)\partial_g \; , \quad \beta(g) = \partial\bar{g}(t, g)/\partial t_{|_{t=1}} \; .$$

The revealing of this symmetry in every particular case is a tricky procedure in theoretical physics.

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What are the distinctive features of RGS in math. physics that distinguish it from the analogues in QFT and MGA?

In group analysis we deal with invariant solutions generated by a group admitted by differential equations. Boundary conditions for these solutions are a priori unspecified.

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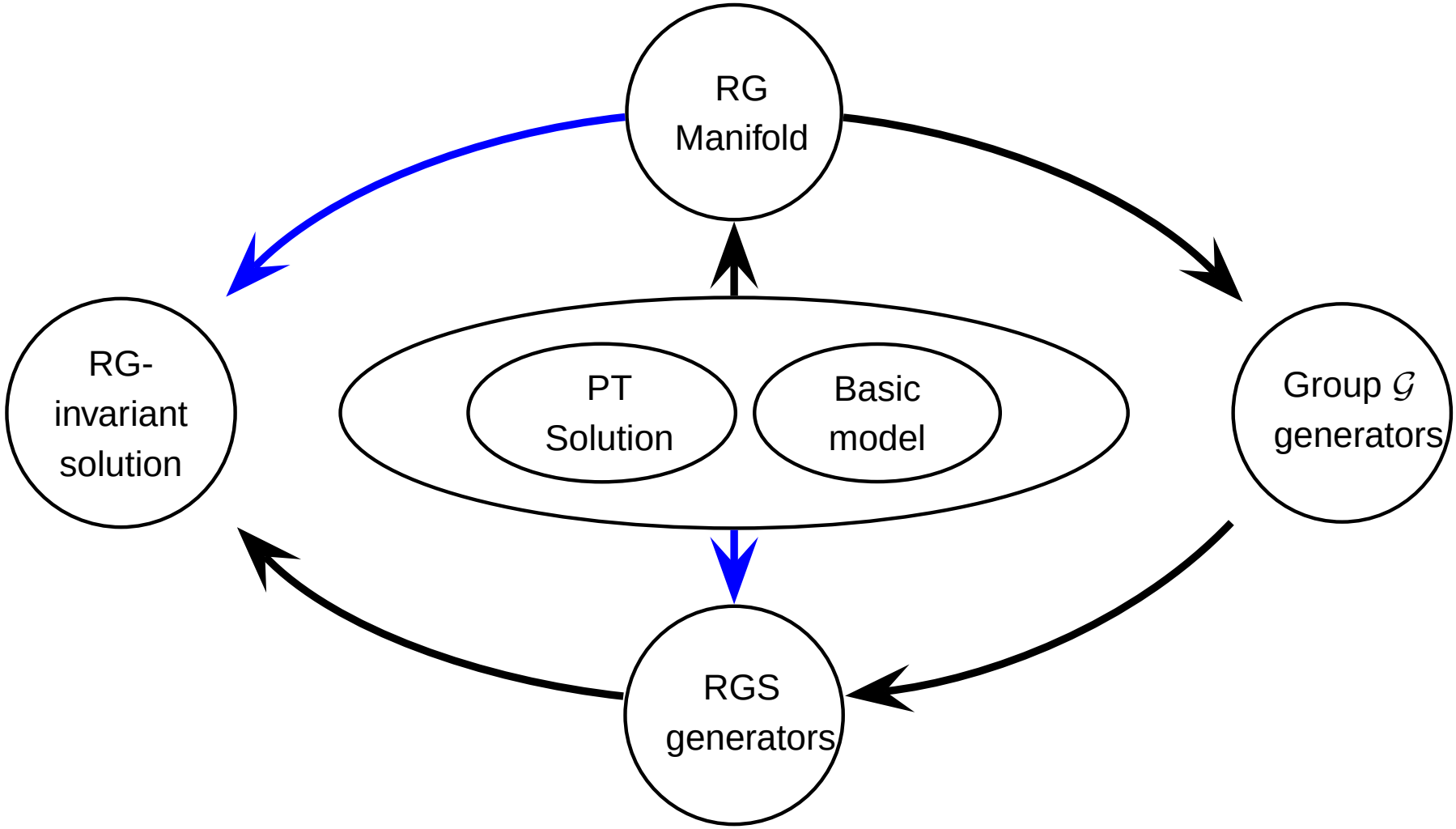
RGS algorithm appears as a synthesis of the theory of invariant solutions and equivalence transformation groups. The concept of RGS helps to realize the close connection of the RGM as formulated by Bogoliubov and Shirkov and modern group analysis.

Sketch of RG algorithm



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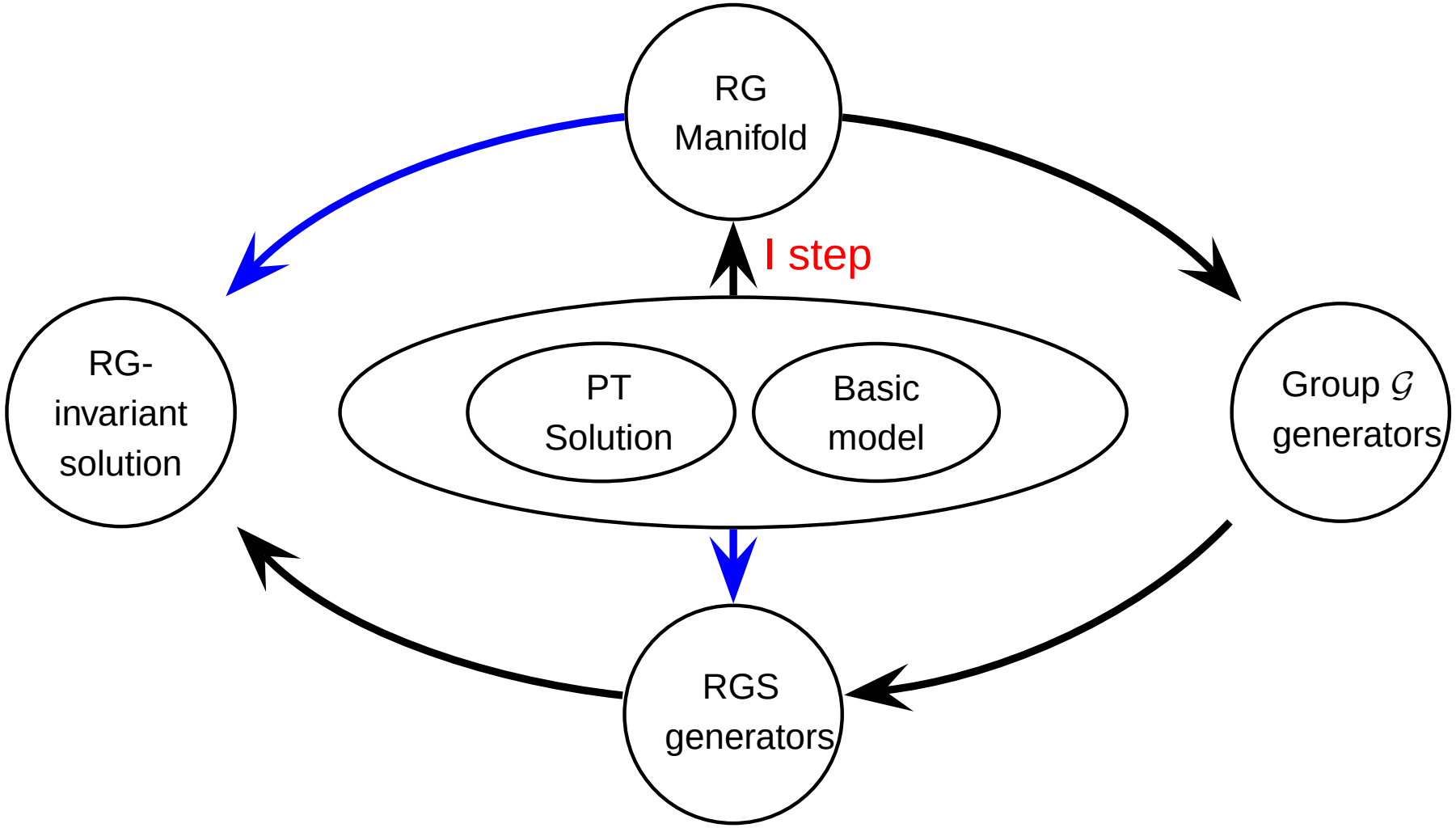

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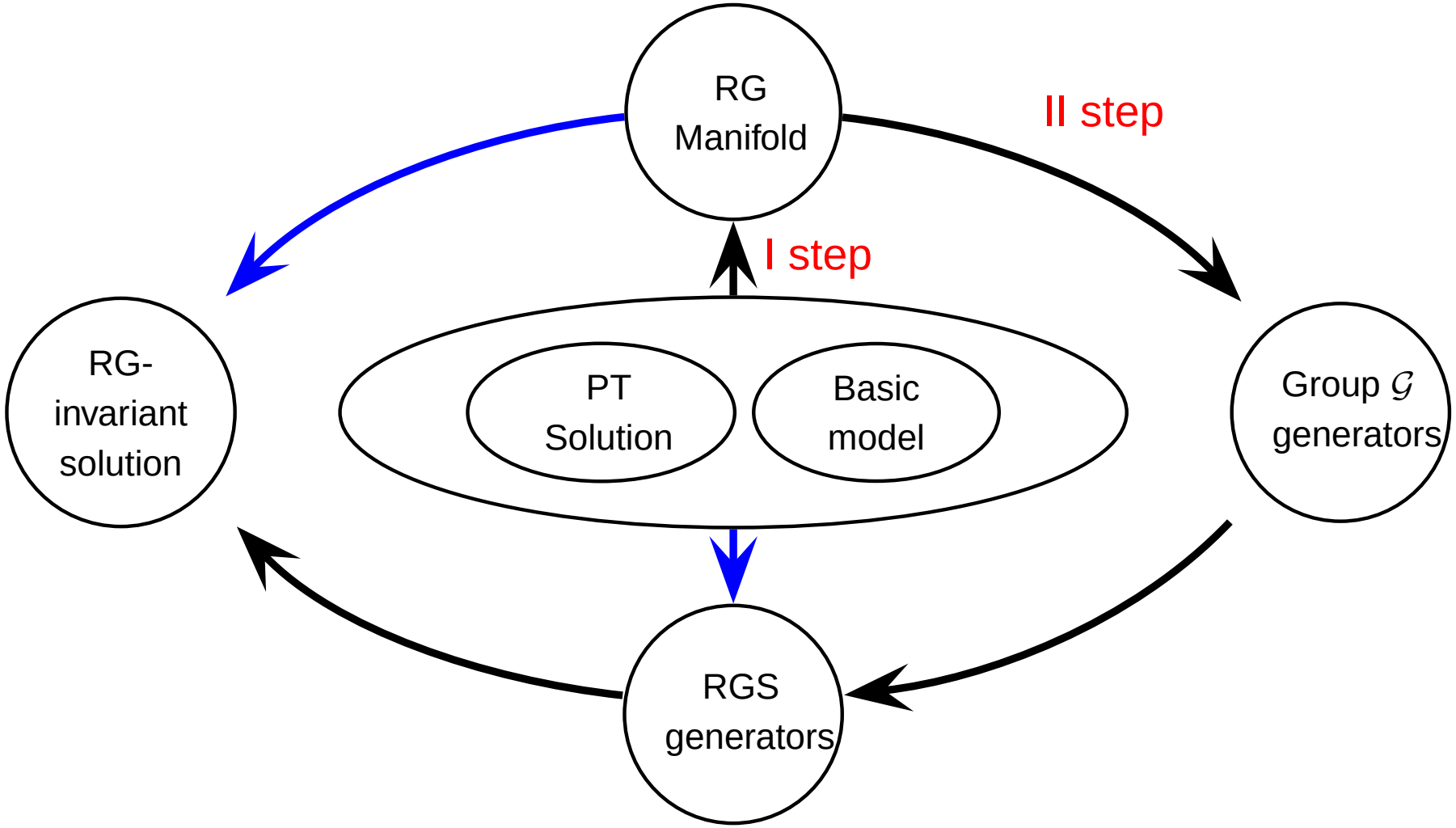

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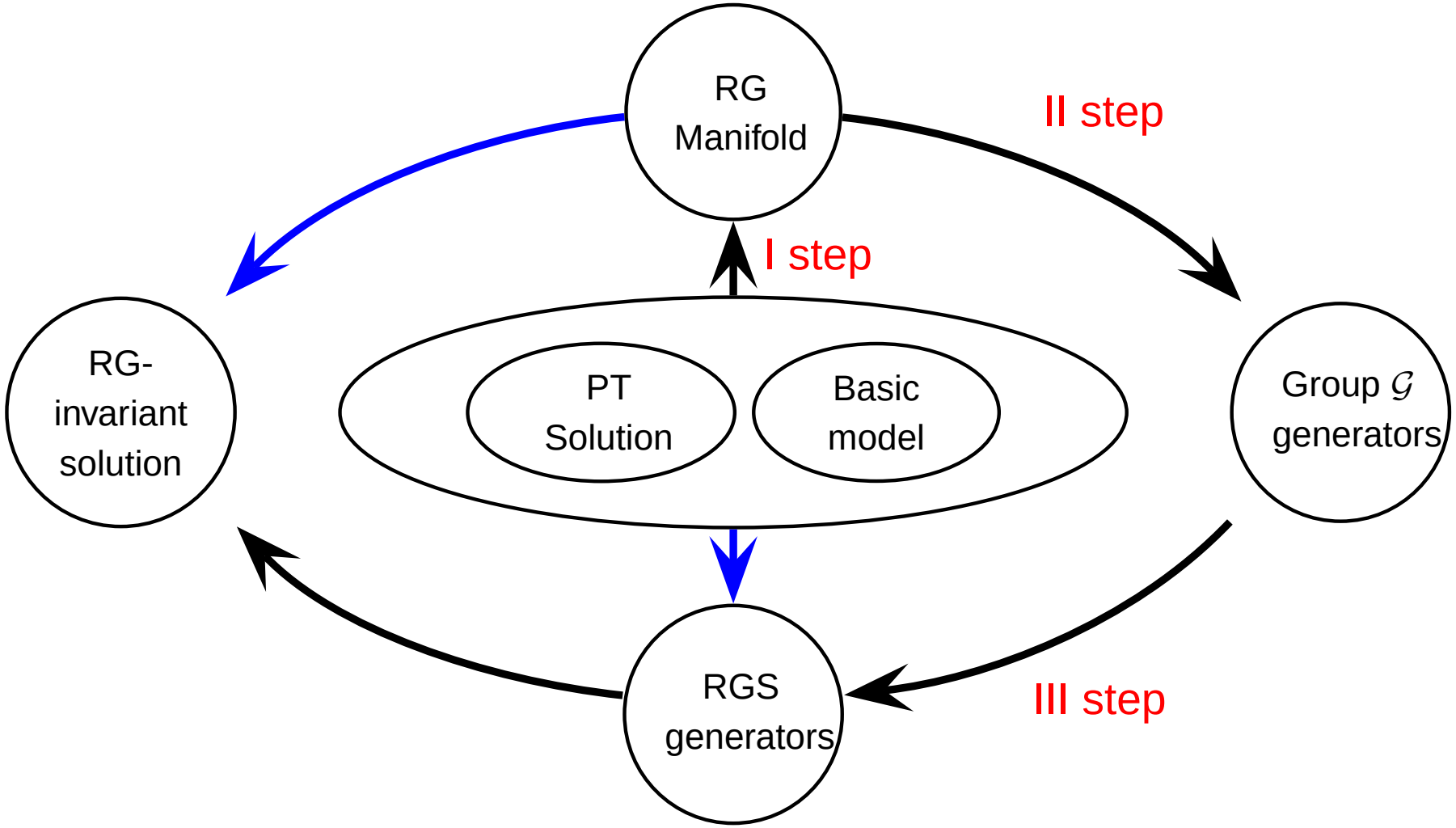

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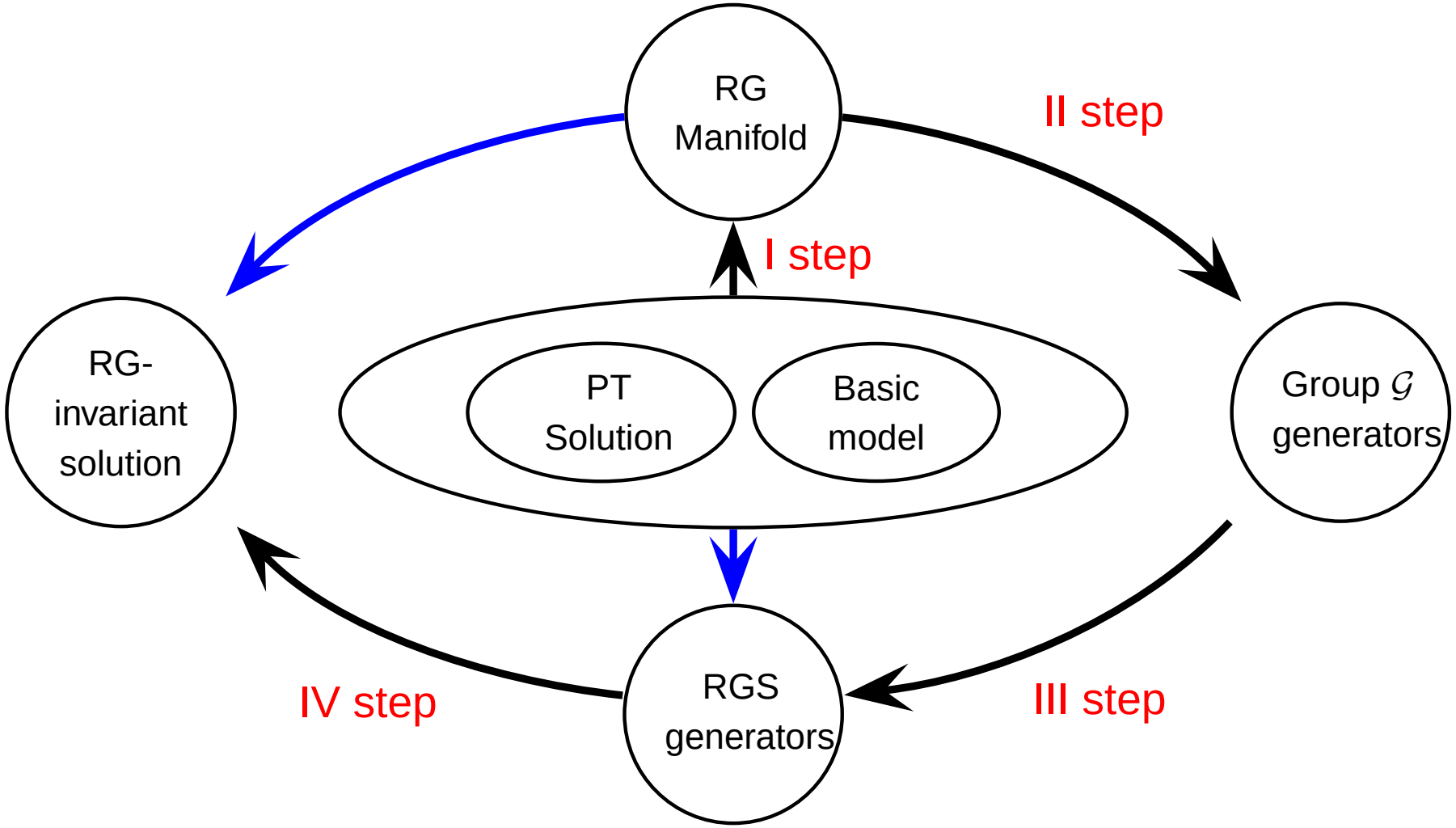

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RGS construction: basic objects

We start with a mathematical model, defined by a system of $\nu \geq 1$ differential and integral equations for the functions $u = \{u^\alpha\}$, $\alpha = 1, \ldots, m$ of $x = \{x^i\}$, $i = 1, \ldots, n$,

$$[E]: \qquad E_\nu(u(x)) = 0\,,$$

with appropriate boundary (initial) conditions. Nonlocal terms in these equations depend on integrals of u. We suppose also that we know some approximate solution, U^α, expressed, say, in the form of the truncated Perturbation Theory (PT) series in powers of some parameter or small distance from the boundary where this solution is given. ➶

Construction of RG-manifold

The key idea of the fist step consists in involving in group transformations the parameters and boundary conditions that define the particular solution of the problem. This is achieved by constructing the special RG manifold $\mathcal{RM}$ that we assume to have the form of s differential equations of the k-th order and q nonlocal relations,

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(k)}) = 0\,, \qquad \sigma = 1, \ldots, s\,,$$
$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(r)}, A(u)) = 0\,, \quad \sigma = 1 + s, \ldots, q + s\,.$$

Here parameters $p = \{p^j\}\,, j = 1, \ldots, l$ are included in $z = \{x, p\}$ and nonlocal variables are given by integral relations

$$A(u) = \int \mathcal{F}(u(z))\mathrm{d}z\,.$$

Construction of $\mathcal{RM}$: possible routines

The particular form of the realization of the first step depends both on the form of input equations and the form of boundary (initial) conditions and is inspired by PT solution as well. We can indicate several possible routines to the problem:

- Extension of the space of variables, involved in group transformations (parameters and differential variables).

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- Extension of the space of variables, involved in group transformations (parameters and differential variables).
- Non-standard form of boundary conditions: utilization of differential constraints, embedding equations (Ambartzumyan, 1942).
- RGS construction for the system of basic equations with small parameter α is based on the *approximate* $\mathcal{RM}$ and utilizes approximate symmetries. ➶

Calculation of the admitted group - 1

The next step is the calculation of the most general symmetry group $\mathcal{G}$, admitted by $\mathcal{RM}$.

For $\mathcal{RM}$ defined by differential equations $F_\sigma = 0,\ \sigma \leq s$ we deal with a local group of transformations that leaves these equations unaltered. The classical Lie algorithm for finding these symmetries consists in constructing tangent vector fields with the generator

$$X = \xi^i \partial_{z^i} + \eta^\alpha \partial_{u^\alpha}\,, \quad \xi^i\,, \eta^\alpha \in \mathcal{A}\,,$$

where coordinates $\xi^i\,,\ \eta^\alpha$ are functions of $\{z^i, u^\alpha\}$ and satisfy the so-called *determining equations* – linear homogeneous partial DEqs.

Return to nonlocal

$$X_{(k)}\, F_\sigma \Big|_{[F_\sigma]} = 0\,, \quad X_{(k)} = X + \ldots + \zeta^\alpha_{i_1 \ldots i_k} \partial_{u^\alpha_{i_1 \ldots i_k}}\,.$$

Calculation of the admitted group - 2

Solving the determining equations gives coordinates ξ^i , η^α, i.e. infinitesimal operators (group generators), which correspond to the admitted vector field and form a Lie algebra with a general element

$$X = \sum_j A^j X_j \,, \quad A^j \text{- arbitrary constants.}$$

Generalization of Lie algorithm $\Rightarrow$ Modern group analysis:

- Approximate symmetries
- Non-classical symmetries
- Non-local symmetries, conditional symmetries
- Higher-order (or Lie-Bäcklund) symmetries

Calculation of group $\mathcal{G}$: generalities

The major obstacle for the application of Lie's infinitesimal techniques to $\mathcal{RM}$ defined by integral equations is that $\mathcal{RM}$ is not *locally* defined in the space of differential functions, hence the crucial idea of splitting of determining equations into over-determined systems, commonly used in the classical Lie group analysis, fails.

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Hence, to apply the RGA to nonlocal models we should clarify two items here:

- Extension of the Lie point symmetry technique for nonlocal $\mathcal{RM}$
- Extension of the RGS operators to nonlocal variables

Calculation of group $\mathcal{G}$: generalities - 2

- Indirect methods
  - Method of moments (Taranov, '74; Bunimovich and Krasnoslobodtsev, '82)
  - Method of boundary-differential equations (Chetverikov and Kudryavtsev, '95)
- Direct methods
  - Grigor'ev and Meleshko, '87; Kovalev, Krivenko, Pustovalov, '92

Canonical representation for X,

$$X \sim Y = X - \xi^i D_i = æ^\alpha \partial_{u^\alpha} + \zeta_i^\alpha \partial_{u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha} + \dots ,$$

$$æ^\alpha = \eta^\alpha - \xi^i u_i^\alpha , \quad \zeta_i^\alpha = D_i(æ^\alpha) , \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(æ^\alpha) .$$

Calculation of group $\mathcal{G}$: generalities - 3

Infinitesimal group transformations with this operator involve only dependent variables u^α and do not change independent variables z^i,

$$u'^{\,\alpha} = u^\alpha + a æ^\alpha + O(a^2)\,, \quad z'^{\,i} = z^i\,.$$

The invariance criterion for F with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator Y – nonlocal determining equations,

$$Y F_\sigma \Big|_{[F_\sigma]} = 0\,, \quad \sigma = 1+s\,,\ldots,\,q+s\,, \quad Y \equiv \int \mathrm{d}z\, æ\,(z)\, \frac{\delta}{\delta u\,(z)}\,.$$

Compare with local

Prolongation on nonlocal variables ➚

To fulfill the procedure of prolongation of any Lie point symmetry generator one should first rewrite this operator in a canonical form and then formally prolong it on the nonlocal variable A

$$Y + æ^A \partial_A \equiv æ \partial_u + æ^A \partial_A \,, \quad A(u) = \int \mathcal{F}(u(z)) \mathrm{d}z \,.$$

The integral relation between æ and $æ^A$ is obtained by applying the generator $Y + æ^A \partial_A$ to the *definition* of A. Substituting æ and calculating integrals obtained gives the desired coordinate $æ^A$

$$æ^A = \int \frac{\delta A(u)}{\delta u(z)} \, æ(z) \, \mathrm{d}z = \int \mathcal{F}_u \, æ(z) \, \mathrm{d}z \,.$$

Generalities - direct method

Algorithm of finding symmetries of nonlocal equations

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Go to Example

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- separating determining equations into local and nonlocal,
- solving local determining equations using a standard Lie algorithm,
- solving nonlocal determining equations using the procedure of variational differentiation.

Go to Example

Restriction the group on a solution

This procedure appears as checking the vanishing condition for the combination of coordinates of the canonical operator on a particular BVP solution $U^\alpha(z)$

$$\sum_j A^j æ_j^\alpha \equiv \sum_j A^j \left(\eta_j^\alpha - \xi_j^i u_i^\alpha\right) \Big|_{u^\alpha = U^\alpha(z)} = 0\,.$$

It transforms the system of DEs for the group invariants into algebraic relations. This procedure has two consequences:

- It gives relations between the A^j thus “combining" different coordinates of the group generators X_j admitted by the $\mathcal{RM}$.
- It eliminates the arbitrariness in the coordinates ξ^i, η^α in the case of an infinite group $\mathcal{G}$.

Restriction the group on a solution

Generally, the restriction procedure reduces the dimension of $\mathcal{G}$. Hence, the general element of the group $\mathcal{G}$ after the fulfillment of a restriction procedure is expressed as a linear combination of the new generators R_i with the coordinates $\hat{\xi}^i$, $\hat{\eta}^\alpha$,

$$X \;\Rightarrow\; R = \sum_j B^j R_j \,, \quad R_j = \hat{\xi}^i_j \partial_{x^i} + \hat{\eta}^\alpha_j \partial_{u^\alpha} \,,$$

with arbitrary constants B^j. We call them renormgroup generators.

This procedure also “fits" boundary conditions into the RG generator by a special choice of coefficients A_j and/or by choosing the particular form of arbitrary functions of the coordinates ξ^i, η^α.

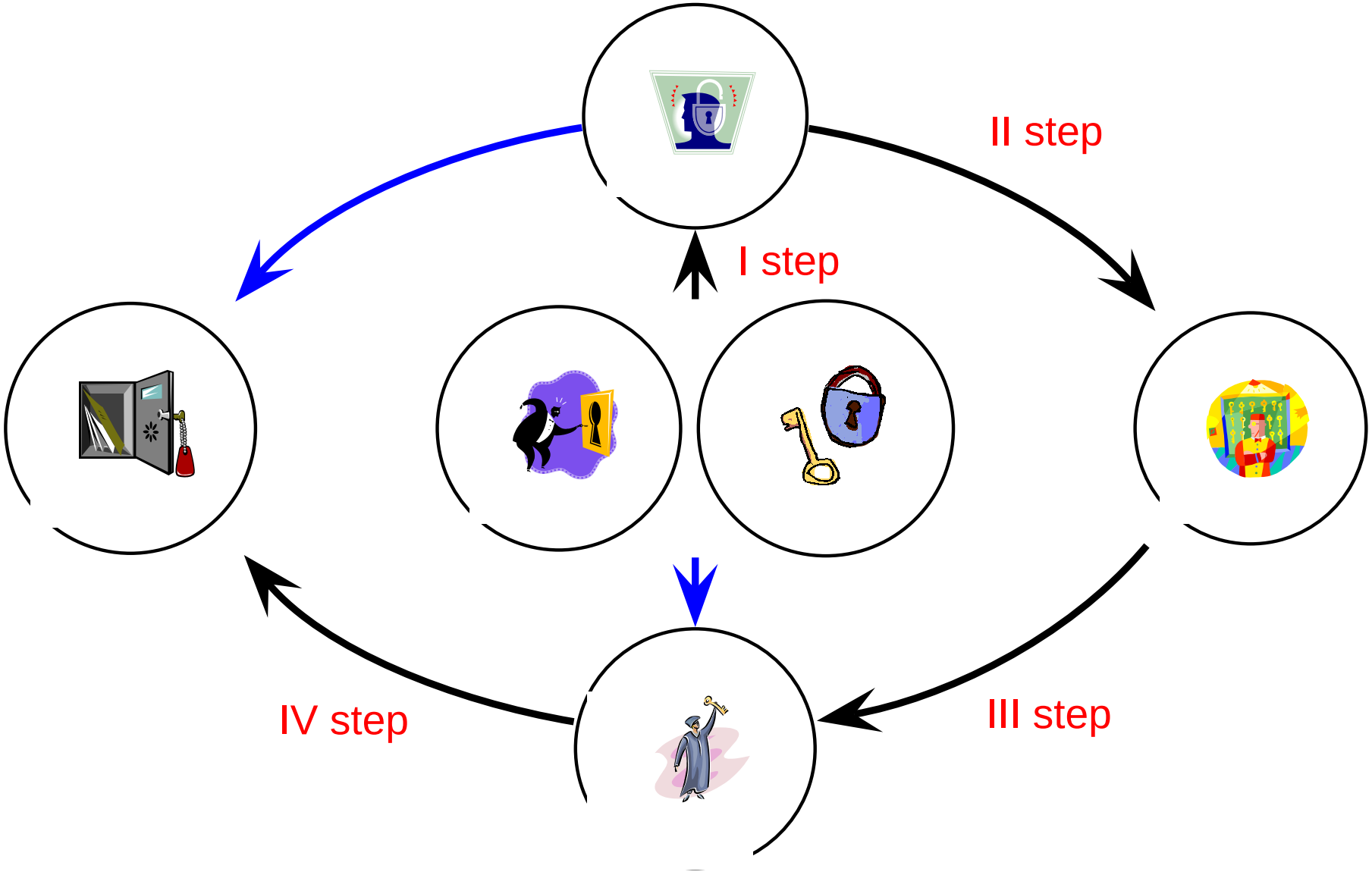
RG invariant solution

Mathematically, this step makes use of the RG=FS *invariance conditions* that are given by a combined system of $\mathcal{RM}$ and the vanishing condition for the linear combination of coordinates of the RG canonical generator

$$\sum_j B^j \hat{æ}_j^\alpha \equiv \sum_j B^j \left(\hat{\eta}_j^\alpha - \hat{\xi}_j^i u_i^\alpha \right) \; = 0 \, .$$

For the *one-parameter Lie point renormgroup*, RG invariance conditions lead to the *first order PDE* that gives rise to the so-called *group invariants* (such as invariant couplings in QFT) which arise as solutions of associated characteristic equations. A general solution of the BVP is now expressed in terms of these invariants.

RGA scheme & clips



Symmetry group for Vlasov-Maxwell equations

☞ Vlasov kinetic equations:

$$f_t^\alpha + \boldsymbol{v} f_{\boldsymbol{r}}^\alpha + \frac{e_\alpha}{m_\alpha \gamma}\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}\boldsymbol{B}] - \frac{1}{c^2}\boldsymbol{v}\,(\boldsymbol{v}\boldsymbol{E})\} f_{\boldsymbol{v}}^\alpha = 0\,; \quad \gamma = \frac{1}{\sqrt{1-(\boldsymbol{v}/c)^2}}\,.$$

Maxwell equations:

$$\boldsymbol{B}_t + c\,\mathrm{rot}\,\boldsymbol{E} = 0\,; \quad \mathrm{div}\,\boldsymbol{E} = 4\pi\rho\,; \quad \boldsymbol{E}_t - c\,\mathrm{rot}\,\boldsymbol{B} + 4\pi\boldsymbol{j} = 0\,; \quad \mathrm{div}\,\boldsymbol{B} = 0\,.$$

Nonlocal material equations:

$$\rho = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{v}\, f^\alpha (\gamma)^5\,, \quad \boldsymbol{j} = \sum_\alpha e_\alpha m_\alpha^3 \int \mathrm{d}\boldsymbol{v}\, f^\alpha (\gamma)^5 \boldsymbol{v}\,.$$

Kovalev,Krivenko,Pustovalov, Diff.Eq.,93; SNMP-95

Symmetry group for Vlasov-Maxwell equations

Vlasov kinetic equations with Lagrange velocity:

Lewak, J.Plasma Phys. 69; Chernikov,Pustovalov, Preprint Lebedev Phys.Inst.(171), 80

$$N_t + \mathrm{div}(N\boldsymbol{V}) = 0\,,$$

$$\boldsymbol{V}_t + (\boldsymbol{V}\nabla)\boldsymbol{V} = \frac{e}{m}\sqrt{1-\left(\frac{\boldsymbol{V}}{c}\right)^2}\left\{\boldsymbol{E} + \frac{1}{c}\,[\boldsymbol{V},\boldsymbol{B}] - \frac{1}{c^2}\boldsymbol{V}\,(\boldsymbol{V},\boldsymbol{E})\right\}.$$

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Nonlocal material equations:

$$\rho = em^3\int \mathrm{d}\boldsymbol{w}\,N\gamma^5\,,\ \boldsymbol{j} = em^3\int \mathrm{d}\boldsymbol{w}\,N\boldsymbol{V}\gamma^5\,,\ \gamma = \frac{1}{\sqrt{1-(\boldsymbol{w}/c)^2}}\,.$$

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$$N_t + \mathrm{div}(N\boldsymbol{V}) = 0\,,$$

$$\boldsymbol{V}_t + (\boldsymbol{V}\nabla)\boldsymbol{V} = \frac{e}{m}\sqrt{1-\left(\frac{\boldsymbol{V}}{c}\right)^2}\left\{\boldsymbol{E} + \frac{1}{c}\left[\boldsymbol{V},\boldsymbol{B}\right] - \frac{1}{c^2}\boldsymbol{V}\left(\boldsymbol{V},\boldsymbol{E}\right)\right\}.$$

Initial conditions: $\boldsymbol{V} = \boldsymbol{w}$, $N = N_0(t_0, \boldsymbol{r}, \boldsymbol{w})$, $\boldsymbol{E} = \boldsymbol{B} = 0$ for $t = t_0$.

Relations between f, N and V:

$$N(t,\boldsymbol{r},\boldsymbol{q}) = f\left(\boldsymbol{p} = \boldsymbol{P}(\boldsymbol{q},\boldsymbol{r},t),\boldsymbol{r},t\right)\det\left(\frac{\partial P_i}{\partial q_j}\right),\ \boldsymbol{V} = c^2\boldsymbol{P}(m^2c^4 + c^2\boldsymbol{P}^2)^{-1/2},$$

$$\boldsymbol{w} = c^2\boldsymbol{q}(m^2c^4 + c^2\boldsymbol{q}^2)^{-1/2}\,, \quad \boldsymbol{v} = c^2\boldsymbol{p}(m^2c^4 + c^2\boldsymbol{p}^2)^{-1/2}.$$

Group generator

The infinitesimal group generator is presented in a standard form:

$$X = \xi^1 \partial_t + \boldsymbol{\xi^2} \partial_{\boldsymbol{r}} + \boldsymbol{\xi^3} \partial_{\boldsymbol{w}} + \eta^1 \partial_N + \eta^2 \partial_{\boldsymbol{V}} + \boldsymbol{\eta^3} \partial_{\boldsymbol{E}} + \boldsymbol{\eta^4} \partial_{\boldsymbol{B}} + \boldsymbol{\eta^5} \partial_{\boldsymbol{j}} + \eta^6 \partial_\rho \,.$$

In the canonical form this operator is written as

$$Y = æ^1 \partial_N + \boldsymbol{æ^2} \partial_{\boldsymbol{V}} + \boldsymbol{æ^3} \partial_{\boldsymbol{E}} + \boldsymbol{æ^4} \partial_{\boldsymbol{B}} + \boldsymbol{æ^5} \partial_{\boldsymbol{j}} + æ^6 \partial_\rho \,,$$

$$æ^1 = \eta^1 - \mathcal{D}N \,, \quad \boldsymbol{æ^2} = \boldsymbol{\eta^2} - \mathcal{D}\boldsymbol{V} \,, \boldsymbol{æ^3} = \boldsymbol{\eta^3} - \mathcal{D}\boldsymbol{E} \,, \quad \boldsymbol{æ^4} = \boldsymbol{\eta^4} - \mathcal{D}\mathbf{B} \,,$$
$$\boldsymbol{æ^5} = \boldsymbol{\eta^5} - \mathcal{D}\boldsymbol{j} \,, \quad æ^6 = \eta^6 - \mathcal{D}\rho \,, \quad \mathcal{D} \equiv \xi^1 \partial_t - (\boldsymbol{\xi^2} \nabla_{\boldsymbol{r}}) - (\boldsymbol{\xi^3} \nabla_{\boldsymbol{w}}) \,.$$

Additional constraints

$$\boldsymbol{E_w} = 0 \,; \quad \boldsymbol{B_w} = 0 \,; \quad \boldsymbol{j_w} = 0 \,; \quad \rho_{\boldsymbol{w}} = 0 \,.$$

Determining equations

Local determining equations

$$D_t\left(æ^1\right) + \left(\boldsymbol{V}, D_{\boldsymbol{r}}\right) æ^1 + N\left(D_{\boldsymbol{r}}, \boldsymbol{æ^2}\right) = 0\,, \quad \Gamma = \left(1 - \left(\boldsymbol{V}/c\right)^2\right)^{-1/2}\,,$$

$$D_t\left(\boldsymbol{æ^2}\right) + \left(\boldsymbol{V}, D_{\boldsymbol{r}}\right) \boldsymbol{æ^2} + \left(\boldsymbol{æ^2}, \nabla\right) \boldsymbol{V} = \frac{e}{m\Gamma}\Bigg\{\boldsymbol{æ^3} + \frac{1}{c}\left(\left[\boldsymbol{V}, \boldsymbol{æ^4}\right] + \left[\boldsymbol{æ^2}, \boldsymbol{B}\right]\right)$$

$$- \frac{1}{c^2}\left(\boldsymbol{æ^2}\left(\boldsymbol{V}, \boldsymbol{E}\right) + \boldsymbol{V}\left(\left(\boldsymbol{æ^2}, \boldsymbol{E}\right) + \left(\boldsymbol{V}, \boldsymbol{æ^2}\right)\right)\right)\Bigg\}$$

$$- \frac{e}{m}\frac{\left(\boldsymbol{V}, \boldsymbol{æ^2}\right)}{c^2}\Gamma\Bigg\{\boldsymbol{E} + \frac{1}{c}\left[\boldsymbol{V}, \boldsymbol{B}\right] - \frac{1}{c^2}\boldsymbol{V}\left(\boldsymbol{V}, \boldsymbol{E}\right)\Bigg\}\,;$$

$$c\left[D_{\boldsymbol{r}}, \boldsymbol{æ^3}\right] + D_t\left(\boldsymbol{æ^4}\right) = 0\,; \quad c\left[D_{\boldsymbol{r}}, \boldsymbol{æ^4}\right] - D_t\left(\boldsymbol{æ^3}\right) = 4\pi\boldsymbol{æ^5}\,;$$

$$\left(D_{\boldsymbol{r}}, \boldsymbol{æ^3}\right) = 4\pi\varkappa^6\,; \quad \left(D_{\boldsymbol{r}}, \boldsymbol{æ^4}\right) = 0\,; \quad D_{\boldsymbol{w}}\left(\boldsymbol{æ^{3,4,5}}\right) = 0\,, \quad D_{\boldsymbol{w}}\left(æ^6\right) = 0\,.$$

Determining equations

Nonlocal determining equations

$$æ^6 - em^4 \int d\boldsymbol{w}\, \gamma^5 æ^1 = 0\,, \quad \boldsymbol{æ^5} - em^3 \int d\boldsymbol{w}\, \gamma^5 \left(æ^1 \boldsymbol{V} + \boldsymbol{æ^2} N\right) = 0\,.$$

Solutions of local determining equations

$$\eta^1 = ((\boldsymbol{b},\, \boldsymbol{r}) + A(\boldsymbol{w}))\, N\,; \quad \boldsymbol{\eta^2} = \boldsymbol{b}\, c^2 - \boldsymbol{V}\, (\boldsymbol{b}, \boldsymbol{V}) + [\boldsymbol{g},\, \boldsymbol{V}]\ ;$$

$$\boldsymbol{\eta^3} = -A_2 \boldsymbol{E} + [\boldsymbol{g},\, \boldsymbol{E}] - c\, [\boldsymbol{b},\, \boldsymbol{B}]\ ; \quad \boldsymbol{\eta^4} = -A_2 \boldsymbol{B} + [\boldsymbol{g},\, \boldsymbol{B}] + c\, [\boldsymbol{b},\, \boldsymbol{E}]\ ;$$

$$\boldsymbol{\eta^5} = -2A_2 \boldsymbol{j} + [\boldsymbol{g},\, \boldsymbol{j}] + c^2 \boldsymbol{b}\, \rho\,; \quad \eta^6 = -2A_2 \rho + (\boldsymbol{b},\, \boldsymbol{j})\ ;$$

$$\xi^1 = A_0 + A_2 t + (\boldsymbol{b},\, \boldsymbol{r})\ ; \quad \boldsymbol{\xi^2} = \boldsymbol{A}_1 + c^2 \boldsymbol{b} t + [\boldsymbol{g},\, \boldsymbol{r}] + A_2 \boldsymbol{r}\,; \quad \boldsymbol{\xi^3} = \boldsymbol{\xi}(\boldsymbol{w})\,,$$

Solutions of nonlocal determining equations

$$A = -2A_2 - 5\frac{(\boldsymbol{w}, \boldsymbol{\xi})}{c^2}\gamma^2 - (\nabla_{\boldsymbol{w}}, \boldsymbol{\xi})\,.$$

Vlasov-Maxwell equations: symmetry group

Lie point symmetry group for Vlasov-Maxwell equations

$$\mathsf{P}_0 = i\partial_t\,; \quad \mathbf{P} = i\partial_{\boldsymbol{r}}\,;$$

$$\mathbf{B} = \boldsymbol{r}\partial_t + c^2 t\partial_{\boldsymbol{r}} - c\,[\boldsymbol{B},\,\partial_{\boldsymbol{E}}] + c\,[\boldsymbol{E},\,\partial_{\boldsymbol{B}}] + c^2\rho\partial_{\boldsymbol{j}} + \boldsymbol{j}\partial_\rho$$
$$+ N\boldsymbol{V}\partial_N + c^2\partial_{\boldsymbol{V}} - \boldsymbol{V}\,(\boldsymbol{V},\,\partial_{\boldsymbol{V}})\,;$$

$$\mathbf{R} = [\boldsymbol{r},\,\partial_{\boldsymbol{r}}] + [\boldsymbol{V},\,\partial_{\boldsymbol{V}}] + [\boldsymbol{E},\,\partial_{\boldsymbol{E}}] + [\boldsymbol{B},\,\partial_{\boldsymbol{B}}] + [\boldsymbol{j},\,\partial_{\boldsymbol{j}}]\,;$$

$$\mathsf{D} = t\partial_t + \boldsymbol{r}\partial_{\boldsymbol{r}} - 2N\partial_N - \boldsymbol{E}\partial_{\boldsymbol{E}} - \boldsymbol{B}\partial_{\boldsymbol{B}} - 2\boldsymbol{j}\partial_{\boldsymbol{j}} - 2\rho\partial_\rho\,;$$

$$X_\infty = \boldsymbol{\xi}\partial_{\boldsymbol{w}} - \left(5\frac{(\boldsymbol{w},\boldsymbol{\xi})}{c^2}\gamma^2 + (\nabla_{\boldsymbol{w}},\boldsymbol{\xi})\right)N\partial_N\,, \quad \boldsymbol{\xi} \equiv \boldsymbol{\xi}(\boldsymbol{w})\,.$$

Poincare group

$$L_{10} =< \mathsf{P}_0,\, \mathbf{P},\, \mathbf{B},\, \mathbf{R} > \,.$$

Applications of RGS to nonlocal models

- Dielectric permittivity of plasma — Example 1
- Laser beam self-focusing — Example 2
- Laser beam refraction in nonlinear medium — Example 3
- Plasma bunch expansion — Example 4

Go to end

Nonlinear dielectric permittivity of plasma

In nonlinear electrodynamics the material equation, i.e. the relation between the induced current density $\boldsymbol{j}(t,\boldsymbol{r})$ and the electric field $\boldsymbol{E}(t,\boldsymbol{r})$, is defined by the dependence of the electric induction $\boldsymbol{D}(t,\boldsymbol{r})$ upon the electric field $\boldsymbol{E}$. In Fourier variables the electric induction is defined as follows

$$\tilde{D}_i(k) = \tilde{E}_i(k) + i\frac{4\pi}{\omega}\tilde{j}_i(k)\,.$$

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For various processes in plasma physics the material equation is expressed as a series expansion in powers of $\boldsymbol{E}$

$$\tilde{j}_i(k) = \sum_l \tilde{j}_i^{(l)}(k)\,, \quad \tilde{j}_i^{(l)}(k) \backsim O(\tilde{\boldsymbol{E}}^l)\,.$$

Due to the temporal and spatial dispersion the relation between the induced current and the electric field is integral (nonlocal). Hence, the material equation has the form of an integral power series in $\boldsymbol{E}$:

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$$\tilde{D}_i(k) = \tilde{E}_i(k) + \sum_l \tilde{j}_i^{(l)}(k) =$$

$$= \varepsilon_{ij}(k)\tilde{E}_j(k) + \sum_{n=2}^{\infty} \int \mathrm{d}k_1 \ldots \mathrm{d}k_n \, \delta(k - k_1 - \ldots - k_n)$$

$$\times \, \varepsilon_{ij_1\ldots j_n}(k_1;\ldots;k_n)\tilde{E}_{j_1}(k_1)\ldots\tilde{E}_{j_n}(k_n)\,, \quad (k) \equiv (\omega, \boldsymbol{k}).$$

Comparison of two parts of this expression gives relations between the nonlinear dielectric permittivity (NDP) tensors of plasma and the current density $\tilde{\boldsymbol{j}}^{(l)}$ of the given order $l \geqslant 2$.

NDP in plasma

In hot plasma NDP are usually obtained by iterating Vlasov-Maxwell equations with the stationary and homogeneous background distribution function $f_0(\boldsymbol{v})$ ☞:

$$f(t,\boldsymbol{r},\boldsymbol{v}) = f_0(\boldsymbol{v}) + \sum_{l\geqslant 1} f^{(l)}(t,\boldsymbol{r},\boldsymbol{v})\,,\ \boldsymbol{j}^{(l)}(t,\boldsymbol{r}) = em\int \mathrm{d}\boldsymbol{v}\boldsymbol{v}(\gamma)^5 f^{(l)}\,.$$

In cold plasma we derive NDP from collisionless hydrodynamic equations

$$N_t + \mathrm{div}(N\boldsymbol{V}) = 0\,,\ \boldsymbol{V}_t + (\boldsymbol{V}\nabla)\boldsymbol{V} = \frac{e}{m}\left\{\boldsymbol{E} + \frac{1}{c}\left[\boldsymbol{V}\boldsymbol{B}\right]\right\},\ \boldsymbol{j} = eN\boldsymbol{V}\,.$$

It is generally taken, that expressions for NDP in hot plasma are of more general type, hence “cold” expressions follow from them in the particular case $f_0(\boldsymbol{v}) = \delta(\boldsymbol{v})$. The use of RGS gives the method of constructing “hot” expression from the “cold” ones. (Kovalev *et al.* RG-91; RG-02)

NDP of plasma

The key idea here is to express the current density $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma as a convolution of a partial current density $\tilde{\boldsymbol{j}}^{(l)}(k, \boldsymbol{w})$ with the equilibrium distribution function $f_0(\boldsymbol{w})$,

$$\tilde{\boldsymbol{j}}^{(l)}(k) = \int \mathrm{d}\boldsymbol{w}\, f_0(\boldsymbol{w}) \tilde{\boldsymbol{j}}^{(l)}(k, \boldsymbol{w})\,,$$

and perform the following steps:

- calculate $\tilde{\boldsymbol{j}}^{(l)}(k, 0)$ in cold plasma using hydrodynamic equations,
- construct $\tilde{\boldsymbol{j}}^{(l)}(k, \boldsymbol{w})$ from $\tilde{\boldsymbol{j}}^{(l)}(k, 0)$ for arbitrary $\boldsymbol{w} \neq 0$,
- integrate $\tilde{\boldsymbol{j}}^{(l)}(k, \boldsymbol{w})$ over $\boldsymbol{w}$ with the weight function $f_0(\boldsymbol{w})$ to get the desired expression for $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma,
- use $\tilde{\boldsymbol{j}}^{(l)}(k)$ to calculate NDP in hot plasma.

NDP of plasma

A transition from the "cold" expression for $\tilde{\boldsymbol{\jmath}}^{(l)}(k,0)$ to the "hot" expression $\tilde{\boldsymbol{\jmath}}^{(l)}(k,\boldsymbol{w})$ is a group transformation that is defined by the corresponding RGS generator

$$\boldsymbol{R} = \boldsymbol{k}\partial_\omega + \partial_{\boldsymbol{w}} - \frac{1}{c}\left[\tilde{\boldsymbol{B}},\, \partial_{\tilde{\boldsymbol{E}}}\right] + \tilde{\varrho}\partial_{\tilde{\boldsymbol{\jmath}}}\,.$$

The finite transformations of this three dimensional RG have the form

$$\omega = \omega' + \boldsymbol{k}'\boldsymbol{w}\,; \quad (\beta_{is}/\omega)\tilde{E}_s = (1/\omega')\tilde{E}_i'\,; \quad \tilde{\varrho} = \tilde{\varrho}'\,; \quad \tilde{\jmath}_i = \beta_{si}\tilde{\jmath}_s'\,;$$
$$\boldsymbol{k} = \boldsymbol{k}'\,; \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}' = (c/\omega')\left[\boldsymbol{k}', \tilde{\boldsymbol{E}}'\right]; \quad \beta_{is} = \delta_{is} + k_i w_s/(\omega - \boldsymbol{k}\boldsymbol{w})\,.$$

For example, in first order $l = 1$ this procedure gives

$$\tilde{\jmath}_a^{(1)}(k,0) = i\frac{e^2 n_{e0}}{m\omega}\tilde{E}_a(k)\,; \Rrightarrow \tilde{\jmath}_i^{(1)}(k,\boldsymbol{w}) = \frac{ie^2 n_{e0}}{m\omega}\beta_{si}\beta_{sa}\tilde{E}_a(k)\,.$$

NDP of plasma

These expressions leads to the one-to-one correspondence between the scalar dielectric permittivity in cold plasma and the corresponding tensor in hot plasma

$$\varepsilon(k) = 1 - \frac{4\pi e^2 n_e}{m\omega^2} \Rrightarrow \varepsilon_{ab}(k) = \delta_{ab} - \frac{4\pi e^2 n_{e0}}{m\omega^2} \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w}) \beta_{sa}\beta_{sb} \,.$$

The proof of this result for the arbitrary order l is straightforward

$$\varepsilon_{ij_1\ldots j_n}(k_1;\ldots;\boldsymbol{k}_n) = \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w}) \frac{\Omega\Omega_1\ldots\Omega_n}{\omega\omega_1\ldots\omega_n} \bar{\varepsilon}_{ab_1\ldots b_n}(\kappa_1;\ldots;\kappa_n)$$
$$\times \beta_{ai}(k)\beta_{b_1j_1}(k_1)\ldots\beta_{b_nj_n}(k_n)\,; \quad \Omega_i \equiv (\omega_i - \boldsymbol{k}_i\boldsymbol{w})\,, \quad \kappa_i \equiv (\Omega_i, \boldsymbol{k}_i).$$

Here $\bar{\varepsilon}$ corresponds to NDP tensor in cold plasma. For $l = 2$ we have,

$$\bar{\varepsilon}_{isj}(\kappa_1;\kappa_2) = -\frac{4\pi i e^3 n_{e0}}{2! m^2 \Omega\Omega_1\Omega_2} \left(\frac{k_i}{\Omega}\,\delta_{js} + \frac{k_{1s}}{\Omega_1}\,\delta_{ij} + \frac{k_{2j}}{\Omega_2}\,\delta_{is} \right) .$$

Go back

Laser beam self-focusing in nonlinear medium

A mathematical model of wave self-focusing is based on the nonlinear Schrödinger (NLS) equation for the complex amplitude of the electric field E of electromagnetic beam:

$$2ik_0\partial_z E + \triangle_\perp E + k_0^2(\epsilon_2/\epsilon_0)E = 0\,, \quad E(0,\mathbf{r}) = E_0(\mathbf{r})\,.$$

It describes a stationary beam propagation in the direction z with an assumption that the wave amplitude scale length along the z-axis is much larger as compared to the characteristic scale in the transversal direction. Here, $k_0 = (\omega/c)\sqrt{\epsilon_0}$ is the wave number, $\triangle_\perp$ is the Laplace operator in the perpendicular plane, $\mathbf{r}$, and ϵ_0 and ϵ_2 are the real parts of linear and nonlinear dielectric permittivities, respectively.
Regular and explosive-like singular solutions have been investigated by using various analytical and numerical methods, which may be attributed to the three categories.

Laser beam self-focusing: different approaches

- Rigorous analytical theories
  - the inverse scattering method Zakharov, Shabat, JETP, 72
  - the classical group analysis Taranov, J.Pl.Phys., 86; Gagnon, Winternitz, J.Phys.A, 88
- Approximate analytical methods
  - the paraxial ray (non-aberrational) approach Talanov, JETP Lett., 65; Akhmanov *et al*.JETP, 66
  - the method of moments Vlasov, Radiophys.,74
  - the variational theory Anderson, Bonnedal, Phys.Fluids, 79
  - modifications of the ISM which employ an asymptotic or perturbation expansions Zakharov-Manakov, JETP, 76; Enns, Rangnekar, Can.J.Phys., 85
- Numerical methods

Laser beam self-focusing: different approaches

None of analytical methods is able to provide an exact solution to the NLS equation for arbitrary boundary conditions. Only some specific BVPs that are far away from practical requirements have been solved analytically so far. There are also a number of contradictions between different theories which have not been resolved yet. A parametric analysis of global self-focusing characteristics versus boundary conditions is still far away from applications.

We present a new, rather general method for finding analytical solutions to the NLS equation by using the RGS approach (Kovalev, Theor.Math.Phys., 99; Kovalev,Bychenkov,Tikhonchuk,Phys.Rev A; 2000.).

Laser beam self-focusing: basic model

We consider a cylindrically symmetric electromagnetic beam incident at $z = 0$ on a homogeneous medium with a cubic nonlinearity, $\epsilon_2(I) = \gamma I$. The electric field, $E = \sqrt{I}\exp(ik_0\Psi)$, is represented in terms of two real functions: the intensity I and the phase Ψ,

$$k_z + kk_r - \alpha I_r - \beta\,\partial_r\left[\frac{1}{r\sqrt{I}}\partial_r(r\partial_r\sqrt{I}\,)\right] = 0\,, \quad I_z + Ik_r + kI_r + \frac{kI}{r} = 0\,.$$

Here, $k = \Psi_r$ is the radial phase derivative, r and z, are normalized by the initial beam radius, r_0, and the intensity is normalized by the on-axis intensity, I_0, at the entrance plane. The boundary conditions, $I(0,r) = I_0 N(r)$, $k(0,r) = -r/R$, assume that the incident beam intensity is an arbitrary function of the radius, $N(0) = 1$, and that the incident beam has a spherical front, $\Psi(0,r) = -r^2/2R$, where R is the radius of the wavefront curvature.

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The basic equations contain two small parameters: the nonlinearity, $\alpha = \epsilon_2(I_0)/2\epsilon_0$, and the diffraction, $\beta = 1/2k_0^2r_0^2$. Both terms can be made arbitrary small if one considers a converging beam and the entrance plane far away from the focal position. Hence, to construct the RGS generator we employ the algorithm of approximate symmetries and expand coordinates of RGS generator in power series over the nonlinearity and diffraction parameters.

Laser beam self-focusing: RGS generator

The RGS generator has the form:

$$R = \left[\left(1-\frac{z}{R}\right)^2 + z^2 S_{\chi\chi}\right]\partial_z + \left[-\frac{r}{R}\left(1-\frac{z}{R}\right) + zS_\chi + kz^2 S_{\chi\chi}\right]\partial_r$$
$$+\left[\frac{r}{R^2}+\frac{k}{R}\left(1-\frac{z}{R}\right)+S_\chi\right]\partial_k + \left[\frac{2I}{R}\left(1-\frac{z}{R}\right) - Iz\left(1+\frac{kz}{r}\right)S_{\chi\chi} - \frac{Iz}{r}S_\chi\right]\partial_I\,,$$

where S depends on $\chi = r - kz$ and two expansion parameters:

$$S(\chi) = \alpha N(\chi) + \frac{\beta}{\chi\sqrt{N(\chi)}}\,\partial_\chi\left[\chi\partial_\chi\sqrt{N(\chi)}\right]\,.$$

RGS generator serves as a tool for finding solutions of the desired boundary value problem. It describes the finite-group transformation that relates the values of the beam intensity and phase for any $z > 0$ to the prescribed data at the boundary.

Laser beam self-focusing: RGS generator

The BVP solution has the form:

$$k(z,r) = \frac{r-\chi}{z}\,, \quad I(z,r) = N(\mu)\left(1-\frac{z}{R}\right)^{-1}\frac{\chi}{r}\,\frac{(S_\chi)^2}{S_{\mu\mu}}\,.$$

The dependence of two additional functions, χ and μ, on z and r is defined by the following relations

$$r = \chi\left(1-\frac{z}{R}\right)\left[1+\frac{2z^2 S_{\chi\chi}}{(1-z/R)^2}\right]\,, \quad S(\mu)-S(\chi) = \frac{z^2(S_\chi)^2}{2(1-z/R)^2}\,.$$

It is important to note that this solution has been derived with no *a priori* assumptions concerning the spatial structure of beam inside the medium.

Laser beam self-focusing: BVP solutions

Example I: For an intensity profile at the boundary that obeys the equation

$$\beta(A_{rr} + (1/r)A_r) + \alpha A^3 - (S_0 + S_2 r^2/2)A = 0\,, \quad N(r) = A^2\,,$$

the function S is given by $S(\chi) = S_0 + S_2\chi^2/2$; in particular, for $S_2 = 0$ we have the "Townes" beam (Townes et al.,PRL,64).
The BVP solution has the form:

$$k = -\frac{r}{R}\,\frac{1 - z/R - S_2 zR}{(1 - z/R)^2 + S_2 z^2}\,, \quad I = N\left(\frac{r}{\sqrt{(1 - z/R)^2 + S_2 z^2}}\right)\frac{1}{(1 - z/R)^2 + S_2 z^2}\,.$$

For $S_2 < 0$ the solution has a singularity on the axes at z_{f1} where the beam intensity goes to infinity

$$I(z, 0) = \frac{z_{f1} z_{f2}}{(z_{f1} - z)\,(z_{f2} - z)}\,, \quad z_{f1,f2} = R/(1 \pm \sqrt{-S_2 R^2})\,.$$

Laser beam self-focusing: BVP solutions

Example II: For a Gaussian intensity profile, $N(r) = \exp\left(-r^2\right)$ we have $S(\chi) = \alpha\, \exp(-\chi^2) + \beta\,(\chi^2 - 2)$, and the beam structure can be written as follows:

$$k(z,r) = \frac{r-\chi}{z}, \quad I(z,r) = \mathrm{e}^{-\mu^2}\left(1-\frac{z}{R}\right)^{-1}\frac{\chi}{r}\,\frac{\beta-\alpha\,\mathrm{e}^{-\chi^2}}{\beta-\alpha\,\mathrm{e}^{-\mu^2}}\,,$$

where the parameters χ and μ are defined by the relations

$$\beta\,(\mu^2-\chi^2) + \alpha\,(\mathrm{e}^{-\mu^2} - \mathrm{e}^{-\chi^2}) = 2z^2\chi^2\,\frac{\left(\beta-\alpha\,\mathrm{e}^{-\chi^2}\right)^2}{(1-z/R)^2}\,,$$

$$r = \rho(z,\chi) \equiv \chi\left(1-\frac{z}{R}\right)\left[1+2z^2\,\frac{\beta-\alpha\,\mathrm{e}^{-\chi^2}}{(1-z/R)^2}\right], \quad \chi = r - kz\,.$$

Laser beam self-focusing: Gaussian beam

The structure of the solution singularity on the beam axis for $\alpha > \beta$ is defined the formulas

$$I(z,0) = \frac{z_{f1} z_{f2}}{(z_{f1} - z)\,(z_{f2} - z)}\,, \quad k(z,0) = 0\,, \quad z_{f1,f2} = \frac{R}{1 \pm R\sqrt{2(\alpha - \beta)}}\,.$$

They describe an explosive intensity growth near the singularity point $z = z_{f1}$. At this point a fractional power radial dependence of I and k near the axis ($r \to 0$) is observed

$$I(z_{f1}, r) = \frac{(R/r)^{2/3}}{\left[2\alpha z_{f1}^2 (R - z_{f1})^2\right]^{1/3}}\,, \quad k(z_{f1}, r) = \frac{r}{z_{f1}} \left[1 - \left(\frac{1 - z_{f1}/R}{2\alpha r^2 z_{f1}^2}\right)^{1/3}\right]\,.$$

Laser beam self-focusing: Gaussian beam

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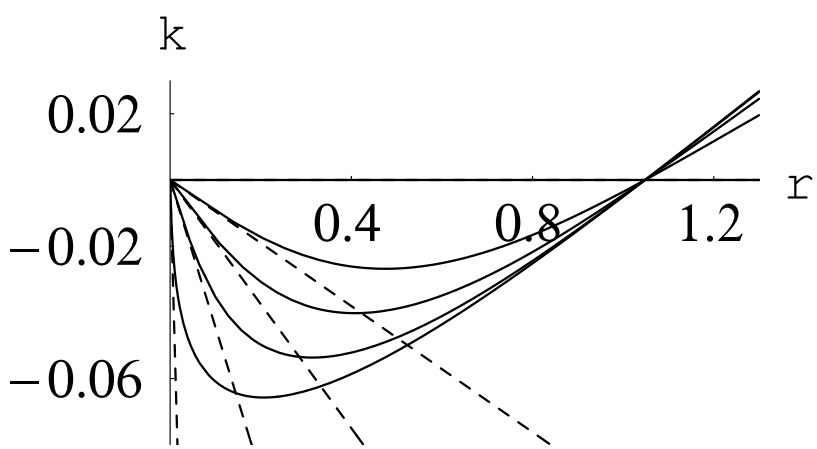
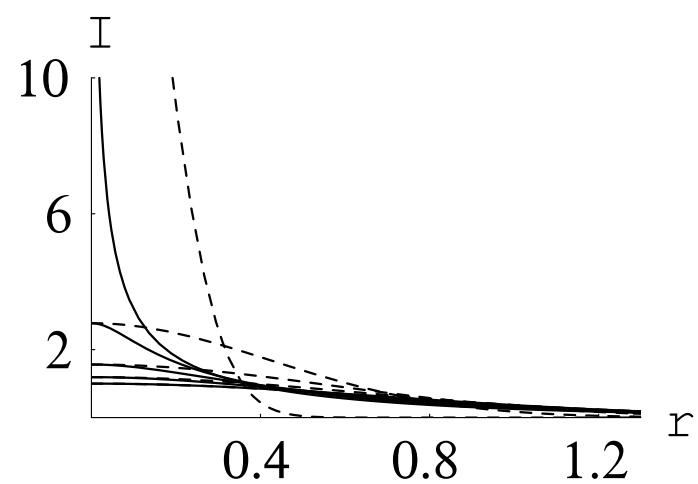
They describe an explosive intensity growth near the singularity point $z = z_{f1}$ that is illustrated below for $z = 0$, 2, 3, 4, 4.9 and $R = \infty$ in a nonlinear medium with $\alpha = 0.03$ and $\beta = 0.01$:





Laser beam self-focusing: global characteristics

Although the RGS method provides a complete BVP solution, it is also instructive to discuss the integral characteristics of a self-focused beam. The beam critical power, $P_c = 2\pi \int Irdr$, defines a minimum beam power, required to create a self-focused channel.

- In the paraxial ray approximation (Akhmanov,66) the critical power of a collimated beam reads $P_c = \pi c^2/\gamma\omega^2$.

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- In the paraxial ray approximation (Akhmanov,66) the critical power of a collimated beam reads $P_c = \pi c^2/\gamma\omega^2$.
- The method of moments (Vlasov,74) predicts four times larger magnitude $P_c^{(m)} = 4P_c = 4\pi c^2/\gamma\omega^2$.

Laser beam self-focusing: global characteristics

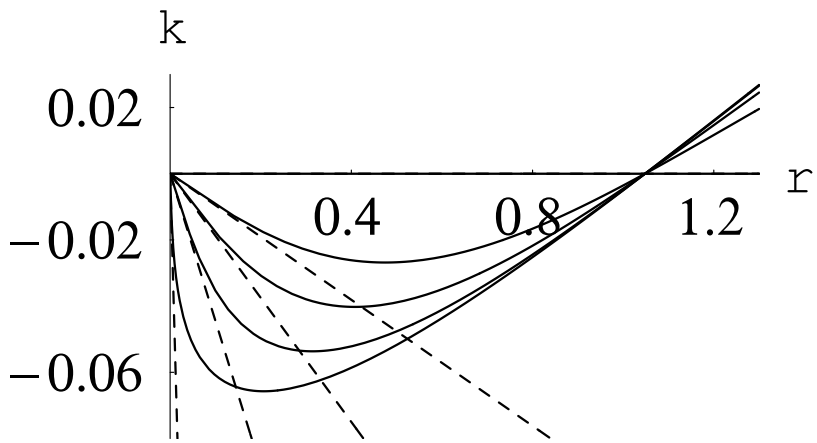
Although the RGS method provides a complete BVP solution, it is also instructive to discuss the integral characteristics of a self-focused beam. The beam critical power, $P_c = 2\pi \int Irdr$, defines a minimum beam power, required to create a self-focused channel.

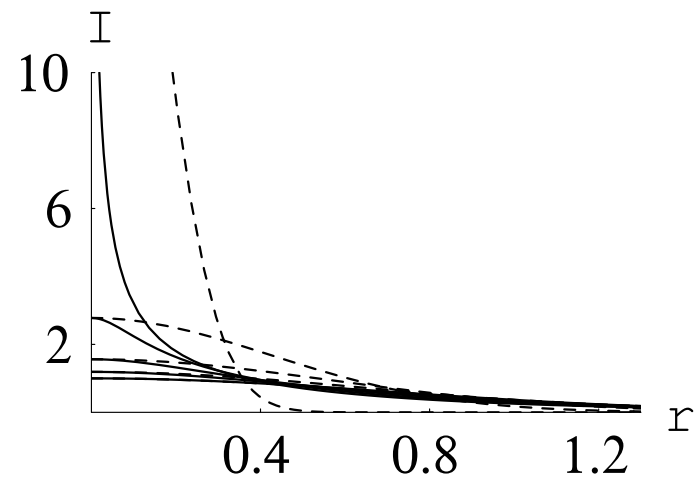
- In the paraxial ray approximation (Akhmanov,66) the critical power of a collimated beam reads $P_c = \pi c^2/\gamma\omega^2$.
- The method of moments (Vlasov,74) predicts four times larger magnitude $P_c^{(m)} = 4P_c = 4\pi c^2/\gamma\omega^2$.

The RGS method resolves a contradiction in the definition of the critical power: the first result defines the power where the singularity on the beam axis shows up, while the second corresponds to the power where the effective beam radius decreases at least at small distances from the entry plane.

Self-focusing: global characteristics-1

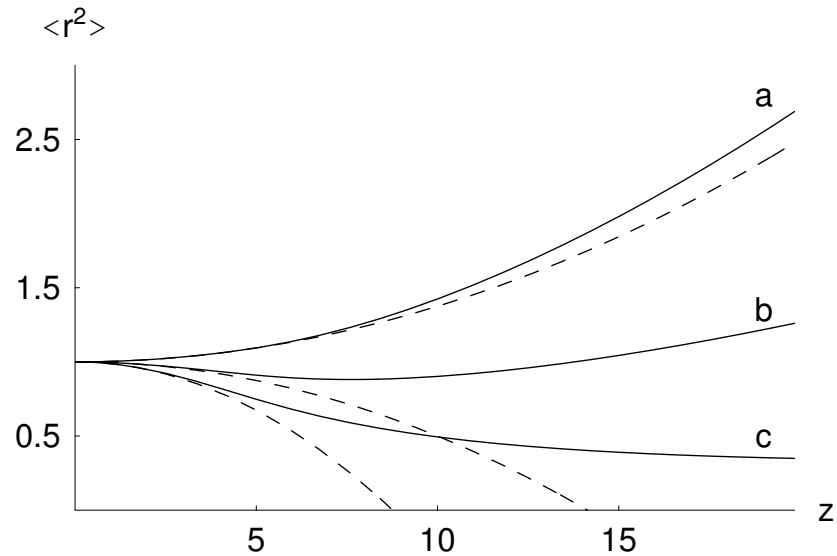
The definition of P_c coincides with the equality $\alpha = \beta$ which is a necessary condition for the singularity to occur. However, the spatial beam structure is quite different from a Gaussian-like beam that the paraxial ray method predicts.





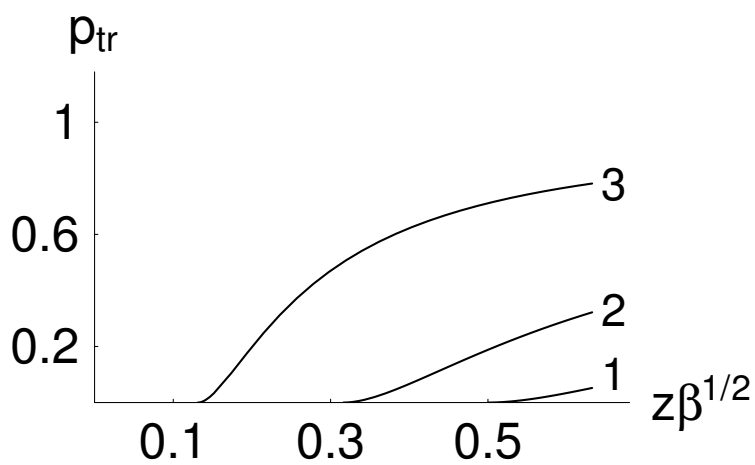
Self-focusing: global characteristics-1

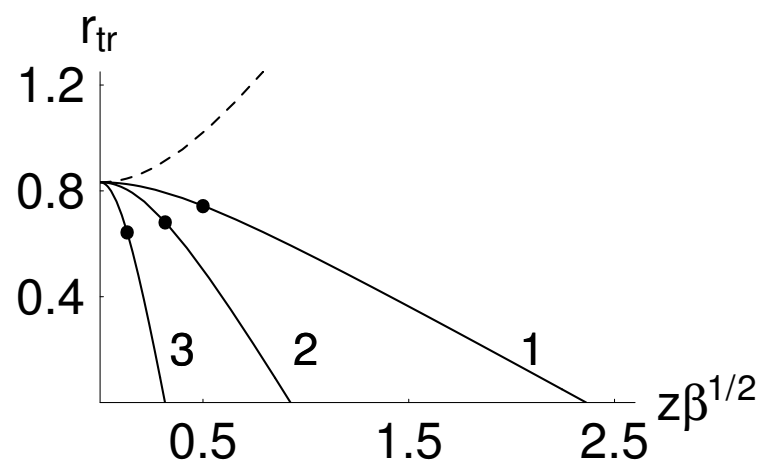
The method of moments identifies the self-focusing threshold as a power where the mean square radius $\langle r^2 \rangle = 2\pi \int I r^3 dr / P(0)$ does not depend on z (for a collimated beam). The dependence of $\langle r^2 \rangle$ on z is shown below in a nonlinear medium with $\alpha = 0.03$ and $\alpha/\beta = 3.2$ (a), 6 (b), and 30 (c). Dashed curves demonstrate the same dependence that follows from the method of moments.



Self-focusing: global characteristics-2

The important characteristics of self-focusing is the amount of power trapped in a singularity and the effective beam radius. The trapped power part is defined as $p_{tr}(z) = 1 - P(z)/P(0)$ where $P(0)$ is the incident beam power and $P(z) = 2\pi \int_0^\infty rI(z,r)\,dr$ and the effective beam radius, $r_{tr}(z)$ is defined as a radius that encircles a half of the incident power, $2\pi \int_{r_{tr}}^\infty Irdr = P(0)/2$. The typical z-dependence of p_{tr} and r_{tr} for a collimated beam in a medium with $\beta = 0.001$ and $\alpha/\beta = 3$ (1), 6 (2), and 30 (3) is shown below.





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Self-focusing: global characteristics-2

The trapped power part, $p_{tr}(z)$, has to be found from the equation

$$\ln \frac{1}{1-p_{tr}} + \frac{\alpha}{\beta}(1-p_{tr}) = \left[1 + \frac{(1-z/R)^2}{2\beta z^2}\right] \left[1 + \ln \frac{2\alpha z^2}{2\beta z^2 + (1-z/R)^2}\right] .$$

For a collimated beam ($R \to \infty$) with $\alpha/\beta = 3$ the trapped power approaches 67% for $z \to \infty$ while for $\alpha/\beta = 30$ the trapped power reaches 97%, i.e. the maximum trapped power equals to the incident beam power with exception of the critical power,

$$p_{tr}^{max} = 1 - \frac{\beta}{\alpha} \equiv 1 - \frac{P_c}{P(0)} .$$

This conclusion is in an apparent contradiction with a heuristic expectation that the critical power should be trapped in a channel while the rest of the incident power could be radiated. Go back

Laser beam refraction in nonlinear medium

The evolution of the laser beam intensity $I(z,x)$ and the eikonal derivative $k(z,x)$ in nonlinear medium ($z>0$) is defined by nonlinear optics equations:

$$k_z + kk_x - \alpha I_x = 0\,,\ I_z + kI_x + Ik_x = \nu Ik/x\,;\ k(0,x) = 0\,,\ I(0,x) = \mathcal{I}(x)\,.$$

Here α is the parameter that define nonlinear refraction, z and x are the coordinates along and transverse to the beam axis, respectively; $\nu = 1$ for cylindrical beam and $\nu = 0$ for the plane beam geometry.
The RGS approach can serve as a regular method of constructing BVP solutions with various boundary conditions. The typical example here is the well-known solution (Akhmanov et al,JETP,66) for the laser beam with the "soliton" intensity distribution at the boundary, $\mathcal{I}(x) = \cosh^{-2}(x)$,

$$k = -2\alpha I z \tanh(x - kz)\,,\ \alpha I^2 z^2 = I\cosh^2(x - kz) - 1\,.$$

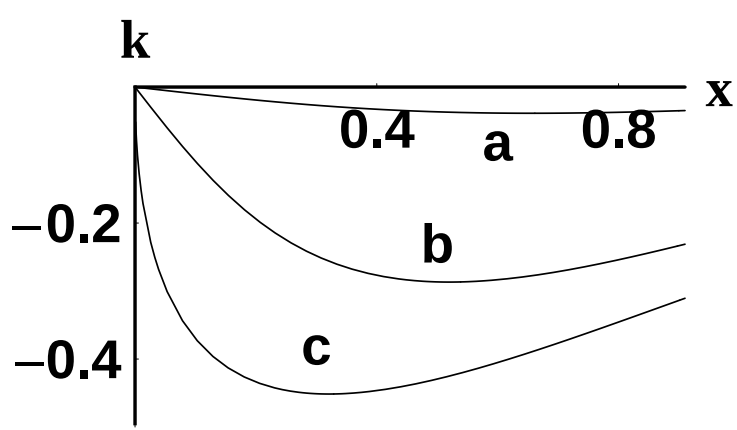
Laser beam refraction: Akhmanov solution

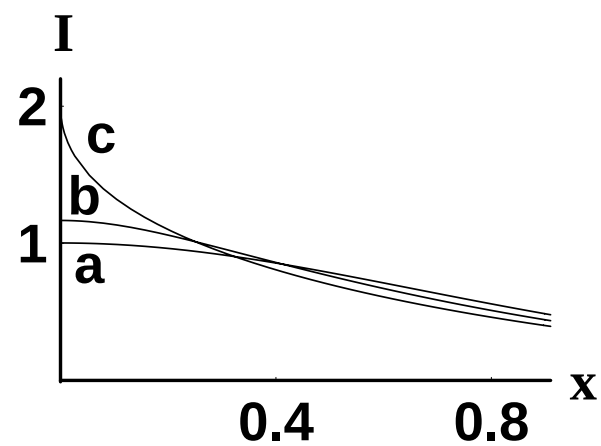
This solution can be obtained in a regular way with the help of the following second order Lie-Bäcklund RGS operator

$$R = \left(2I(1-I)\tau_{II} - I\tau_I - 2I\mathcal{K}(\chi_I + I\chi_{II}) + \frac{\alpha}{2} I\mathcal{K}^2\tau_{II}\right)\partial_\tau$$
$$+ \left(2I(1-I)\chi_{II} + (2-3I)\chi_I + \alpha\mathcal{K}\left(2I\tau_{II} + \tau_I\right) + \frac{\alpha\mathcal{K}^2}{2}(I\chi_{II} + \chi_I)\right)\partial_\chi\,,$$

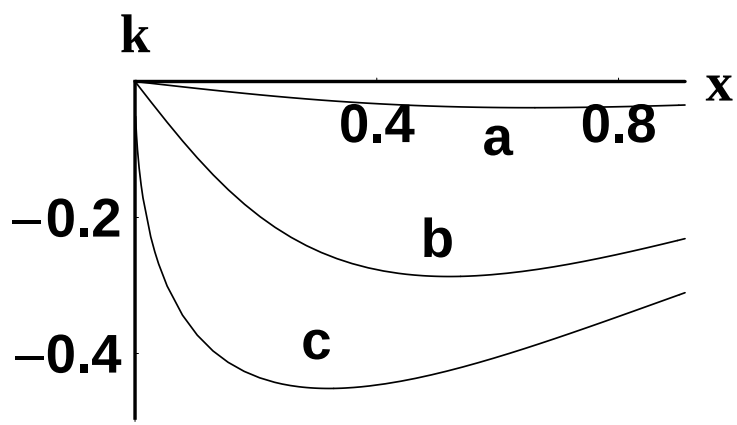
where $\chi = x - kz$, $\tau = Iz$ and $\mathcal{K} = k/\alpha$.

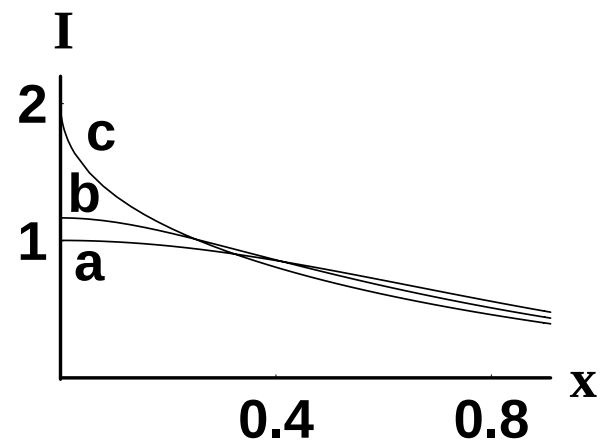
The typical behavior of k and I is illustrated below for $z = 0.05$ (a), 0.35 (b) and 0.5 (c) in a nonlinear medium with $\alpha = 1$.





Laser beam refraction: Akhmanov solution





The appearance of the singularity in the solution is obtained using the reduced description via two solution functionals, namely laser beam intensity, $I^0(z) \equiv I(z,0)$, and the second derivative of the eikonal, $W^0(z) \equiv k_x(z,0)$, on the beam axis, that may be formally introduces as

$$I^0(z) = \int \mathrm{d}x\, \delta(x) I(z,x)\,,\; W^0(z) = \int \mathrm{d}x\, \delta(x) k_x(z,x)\,,\; I^0(0) = 1,\; W^0(0) = 0\,.$$

Parabolic laser beam

For the cylindrical laser beam ($\nu = 1$) with the parabolic intensity distribution, $\mathcal{I}(x) = 1 - x^2$, the RGS generator is given by

$$R^{par} = \left(1 - 2\alpha z^2\right)\partial_z - 2\alpha z x \partial_x - 2\alpha\left(x - vz\right)\partial_k + 4\alpha I z \partial_I \,.$$

Prolongation on "nonlocal" variables I^0 and W^0 yields

$$R = \left(1 - 2\alpha z^2\right)\partial_z + 4\alpha I^0 z \partial_{I^0} - 2\alpha(1 - 2zW^0)\partial_{W^0} \,.$$

Solving Lie equations gives the behavior of $I^0(z)$ and $W^0(z)$

$$I^0 = \frac{1}{1 - 2\alpha z^2}\,, \quad W^0 = -\frac{2\alpha z}{1 - 2\alpha z^2}\,,$$

starting from the boundary of the nonlinear medium up to the point of the solution singularity, $z_{sing} = 1/\sqrt{2\alpha}$, where both the beam intensity and the eikonal derivative go to infinity. (Kovalev TMP,97;Nonlin.Dyn.,2000.)

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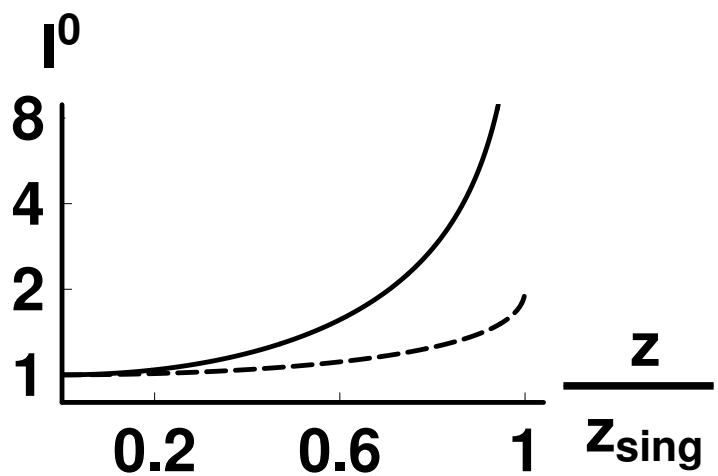
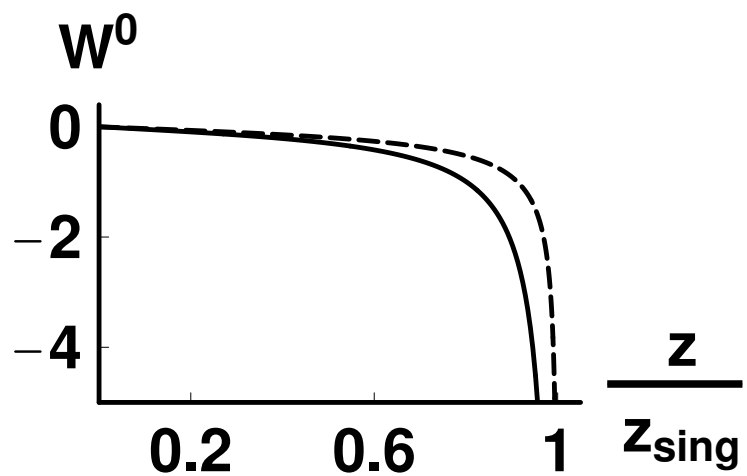
The typical behavior of I^0 and W^0 is presented below (solid lines) from the boundary up to the singularity point





Soliton-type laser beam

The same, though more complicated procedure can be fulfilled in case when the RG generator is presented by Lie-Bäcklund symmetry generator, e.g. when we have the “soliton” profile at the boundary, $\mathcal{I}(x) = \cosh^{-2}(x)$, (Kovalev *et. al*, Diff.Eq.93; SNMP-95; TMP,97; Nonlin.Dyn.,2000)

$$
\begin{aligned}
R^{sol} = \Big\{ & \frac{I}{(Ik_x^2 + \alpha I_x^2)^2} \Big[\Big(\frac{1}{2} \left(Ik_x^2 - \alpha I_x^2 \right) \left(k^2 + 4\alpha(1-I) \right) + 4\alpha k I I_x k_x \Big) v_{xx} \\
& + \left(2\alpha k \left(\alpha I_x^2 - Ik_x^2 \right) + \alpha k_x I_x \left(k^2 + 4\alpha(1-I) \right) \right) \Big(I_{xx} - \frac{I_x^2}{2I} \Big) \Big] - k(1 - zk_x) - \alpha z I_x \Big\} \partial_k \\
& + \Big\{ \frac{I}{(Ik_x^2 + \alpha I_x^2)^2} \Big[\Big(\frac{1}{2} \left(Ik_x^2 - \alpha I_x^2 \right) \left(k^2 + 4\alpha(1-I) \right) + 4\alpha k I k_x I_x \Big) \Big(I_{xx} - \frac{I_x^2}{2I} \Big) \\
& - \left[2k \left(\alpha I_x^2 - Ik_x^2 \right) + k_x I_x \left(k^2 + 4\alpha(1-I) \right) \right] I k_{xx} + \frac{1}{4I} (Ik_x^2 + \alpha I_x^2) \left[4\alpha I_x^2 \right. \\
& \left. + (I_x k - 2Ik_x)^2 \right] \Big] - I(2 - zk_x) + zkI_x \Big\} \partial_I \, .
\end{aligned}
$$

Prolongation on nonlocal variables, gives the RGS generator

Soliton-type laser beam

$$R = R^{sol} + \left(4 - 5I^0 - zI_z^0 + 2(I^0 - 1)\frac{I^0 I_{zz}^0}{(I_z^0)^2}\right)\partial_{I^0}$$
$$+ \left(\frac{I_z^0}{I^0} + z\frac{I_{zz}^0}{I^0} - z\left(\frac{I_z^0}{I^0}\right)^2 - 2(I^0 - 1)\left[\frac{I_{zzz}^0}{(I_z^0)^2} + 2\frac{I_z^0}{(I^0)^2} - 2\frac{(I_{zz}^0)^2}{(I_z^0)^3}\right]\right)\partial_{W^0}\,.$$

Utilizing RG invariance conditions in view of the additional constraint, $(I_z^0/\sqrt{I^0 - 1})_{|z\to 0} = 2\sqrt{\alpha}$, yields the behavior of I^0 and W^0,

$$z = \frac{\sqrt{I^0 - 1}}{\sqrt{\alpha}I^0}\,, \quad W^0 = -\frac{2\alpha z I^0}{1 - 2\alpha z^2 I^0}\,,$$

from the boundary $z = 0$ up to the singularity point $z_{sing} = 1/2\sqrt{\alpha}$, where $I^0 = 2$.

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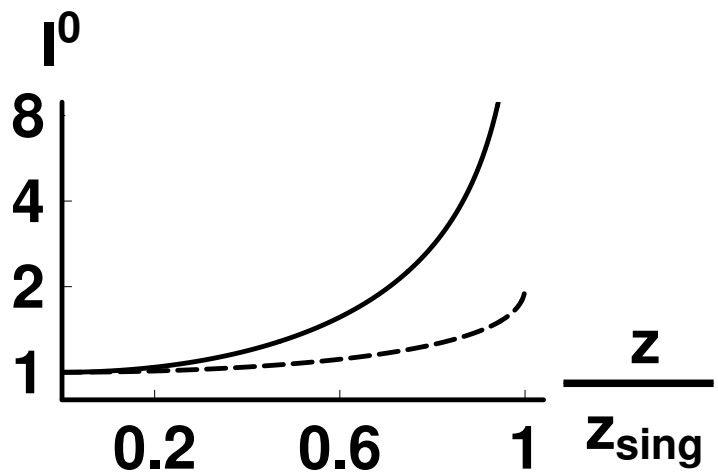
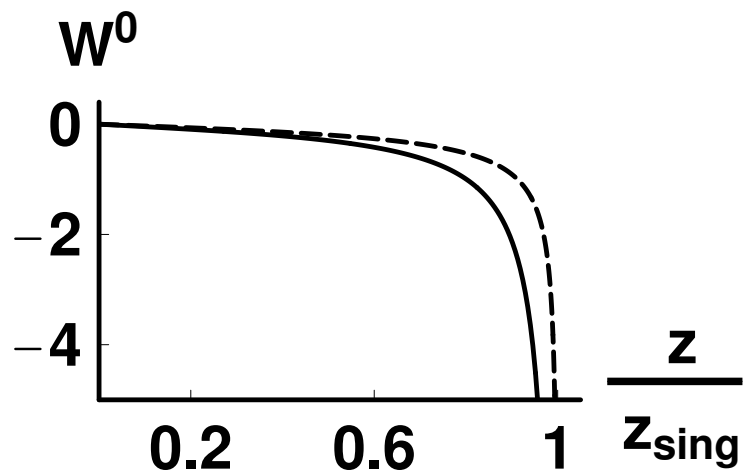
Soliton-type laser beam

$$R = R^{sol} + \left(4 - 5I^0 - zI_z^0 + 2(I^0 - 1)\frac{I^0 I_{zz}^0}{(I_z^0)^2}\right)\partial_{I^0}$$
$$+ \left(\frac{I_z^0}{I^0} + z\frac{I_{zz}^0}{I^0} - z\left(\frac{I_z^0}{I^0}\right)^2 - 2(I^0 - 1)\left[\frac{I_{zzz}^0}{(I_z^0)^2} + 2\frac{I_z^0}{(I^0)^2} - 2\frac{(I_{zz}^0)^2}{(I_z^0)^3}\right]\right)\partial_{W^0}\,.$$

The typical behavior of I^0 and W^0 is presented below (dashed lines) from the boundary up to the singularity point





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Expansion of a plasma bunch

We consider the 1-D expansion of a plasma bunch, which is inhomogeneous in x. The basic system include kinetic equations for particle distribution functions f^α,

$$f_t^\alpha + v f_x^\alpha + (e_\alpha/m_\alpha) E(t,x) f_v^\alpha = 0\,, \quad f^\alpha\big|_{t=0} = f_0^\alpha(x,v)\,,$$

with the additional quasi-neutrality conditions

$$\int \mathrm{d}v \sum_\alpha e_\alpha f^\alpha = 0\,, \quad \int \mathrm{d}v\, v \sum_\alpha e_\alpha f^\alpha = 0\,.$$

Electric field is expressed in terms of moments of distribution functions.

$$E(t,x) = \int \mathrm{d}v\, v^2\, \partial_x \sum_\alpha e_\alpha f^\alpha \left\{ \int \mathrm{d}v \sum_\alpha \frac{e_\alpha^2}{m_\alpha} f^\alpha \right\}^{-1}.$$

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RGS method allows to derive an entire class of solutions to the Cauchy problem for different initial distributions of the particles. (Kovalev *et al.* JETP,02; PRL,03.)

Expansion of a plasma bunch: RG generator

The RGS generator is the linear combination of time translations and the projective operator

$$R = (1 + \Omega^2 t^2)\partial_t + \Omega^2 t x \partial_x + \Omega^2 (x - vt)\partial_v \, .$$

This operator is the only which selects the spatially symmetric initial DFs with zero mean velocity. The value Ω can be treated as the ratio of the ion acoustic velocity to the gradient length L_0.
Distribution functions are invariants of the RG transformations, i.e.

$$f^\alpha = f_0^\alpha(I^{(\alpha)}) \, , \; I^{(\alpha)} = \frac{1}{2}\left(v^2 + \Omega^2 (x - vt)^2\right) + \frac{e_\alpha}{m_\alpha}\Phi_0(x') \, ,$$

The dependence of Φ_0 on self-similar variable $x' = x/\sqrt{1 + \Omega^2 t^2}$ is defined by quasi-neutrality conditions.

Expansion of a plasma bunch: RG generator

For practical applications rough integral characteristics, such as partial ion density, $n_q(t,r)$, might be more useful,

$$n^q(t,x) = \int\limits_{-\infty}^{\infty} \mathrm{d}v f^q(t,x,v)\,.$$

One may consider $n^q(t,x)$ as the linear functional of f^q and prolong the RG generator to get the following operator

$$R = (1+\Omega^2 t^2)\partial_t + \Omega^2 t x \partial_x + \Omega^2 t n^q \partial_{n^q}\,.$$

Finite group transformations defined by this operator yield

$$n^q = \frac{n_{q0}}{\sqrt{1+\Omega^2 t^2}}\mathcal{N}_q\left(\frac{x}{\sqrt{1+\Omega^2 t^2}}\right)\,, \quad \mathcal{N}_q = \int\limits_{-\infty}^{\infty} \mathrm{d}v f_0^q\,.$$

Expansion of a plasma bunch: illustration

For an expansion of a plasma bunch with two different ion species having Maxwellian distribution functions and electrons obeying a two-temperature Maxwellian distribution function with densities and temperatures of the cold and hot components n_{c0} and n_{h0} ($n_{c0} + n_{h0} = Zn_{i0}$) and T_e and T_h, respectively, the solution to the Cauchy problem reads:

$$f^e = \frac{n_{c0}}{\sqrt{2\pi} v_{Tc}} \exp\left(-\frac{I^{(c)}}{v_{Tc}^2}\right) + \frac{n_{h0}}{\sqrt{2\pi} v_{Th}} \exp\left(-\frac{I^{(h)}}{v_{Th}^2}\right), \quad v_{T\alpha}^2 = \frac{T_\alpha}{m_\alpha},$$

$$f^q = \frac{n_{q0}}{\sqrt{2\pi} v_{Tq}} \exp\left(-\frac{I^{(q)}}{v_{Tq}^2}\right), \; u = xt\frac{\Omega^2}{(1+\Omega^2 t^2)}, \; U = x\frac{\Omega}{\sqrt{1+\Omega^2 t^2}},$$

$$\frac{I^{(c)}}{v_{Tc}^2} = \mathcal{E} + \frac{(1+\Omega^2 t^2)}{2v_{Tc}^2}(v-u)^2, \quad \frac{I^{(h)}}{v_{T_h}^2} = \mathcal{E}\frac{T_c}{T_h} + \frac{(1+\Omega^2 t^2)}{2v_{Th}^2}(v-u)^2,$$

$$\frac{I^{(q)}}{v_{Tq}^2} = -\mathcal{E}\left(\frac{Z_q T_{c0}}{T_{q0}}\right) + \frac{U^2}{2v_{T_q}^2}\left(1 + \frac{Z_q m_e}{m_q}\right) + \frac{(1+\Omega^2 t^2)}{2v_{T_q}^2}(v-u)^2, \; q = 1, 2.$$

Expansion of a plasma bunch: illustration

Here the function $\mathcal{E}$ satisfy the equation,

$$n_{c0} = \sum_{q=1,2} Z_q n_{q0} \exp\left[\left(1 + \frac{Z_q T_c}{T_q}\right)\mathcal{E} - \frac{U^2}{2v_{Tq}^2}\left(1 + \frac{Z_q m_e}{m_q}\right)\right] - n_{h0} \exp\left[\left(1 - \frac{T_c}{T_h}\right)\mathcal{E}\right].$$

For the specified initial distribution functions the ion density distributions $\mathcal{N}_q$ in carbon-proton plasma have the form

$$\mathcal{N}_q = \exp\left[\mathcal{E}\left(\frac{Z_q T_{c0}}{T_{q0}}\right) - \frac{U^2}{2v_{T_q}^2}\left(1 + \frac{Z_q m_e}{m_q}\right)\right], \quad q = 1, 2\,.$$

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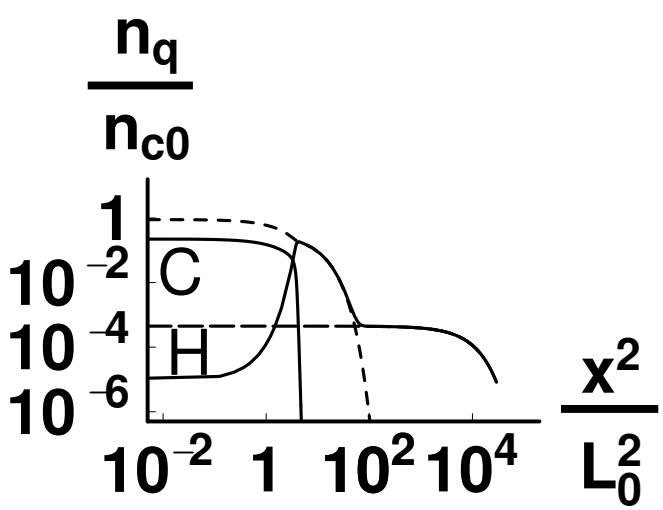
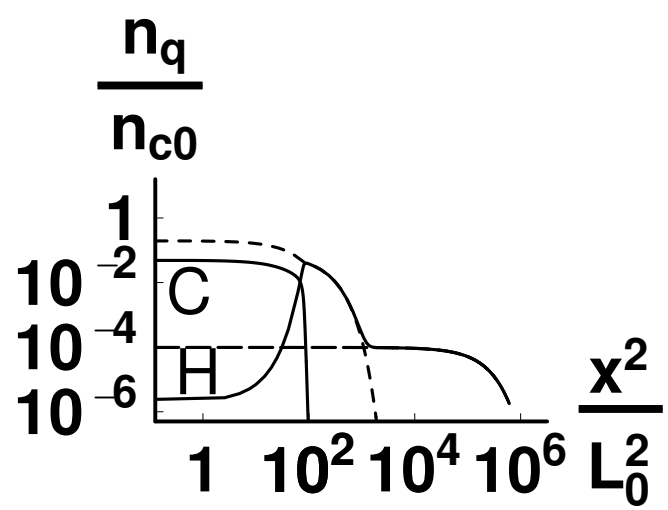
Expansion of a plasma bunch: illustration

Here the function $\mathcal{E}$ satisfy the equation,

$$n_{c0} = \sum_{q=1,2} Z_q n_{q0} \exp\left[\left(1+\frac{Z_q T_c}{T_q}\right)\mathcal{E} - \frac{U^2}{2v_{Tq}^2}\left(1+\frac{Z_q m_e}{m_q}\right)\right] - n_{h0}\exp\left[\left(1-\frac{T_c}{T_h}\right)\mathcal{E}\right].$$

For the specified initial distribution functions the ion density distributions n_q in carbon-proton plasma are presented below for two different moments: $t_1 = 0.5L_0/c_s$ and $t_2 = 10t_1$.





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Conclusion

We hope that this report serves as an illustration of the universality of the RGS algorithm in application both to models based on DE and nonlocal (integral) equations. The possibility to prolong the symmetry on solution functionals enables to investigate the behavior of solution characteristics even in the case when the explicit form of the solution is not known. The results presented gives the promise for the progress in this field and can give rise to many potential applications. ➦