Fourier Based Analysis of Classical, Modulated and Complex Interferograms

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3. Fourier Method of Analysis (Stationary Objects)
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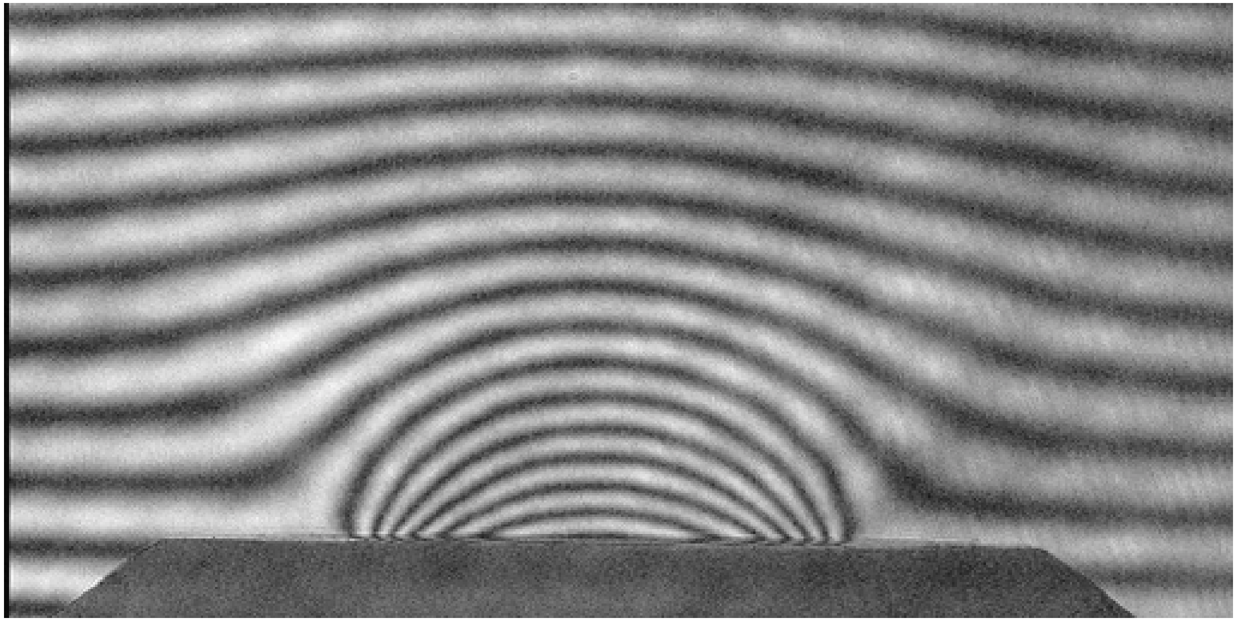
1. Examples of Various Interferograms

Experimental Interferogram

Classical

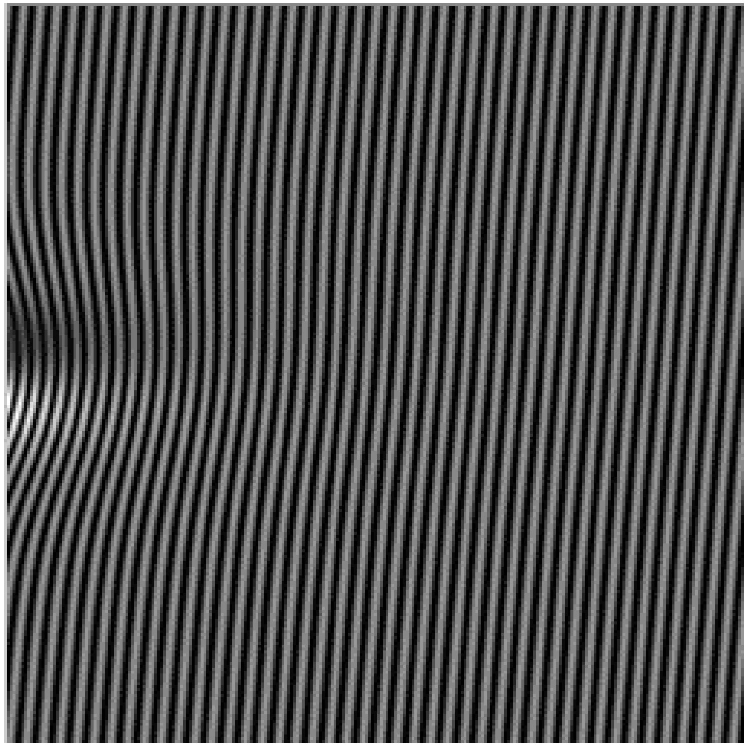
1 Degree of Freedom



Synthetic Interferogram

Modulated

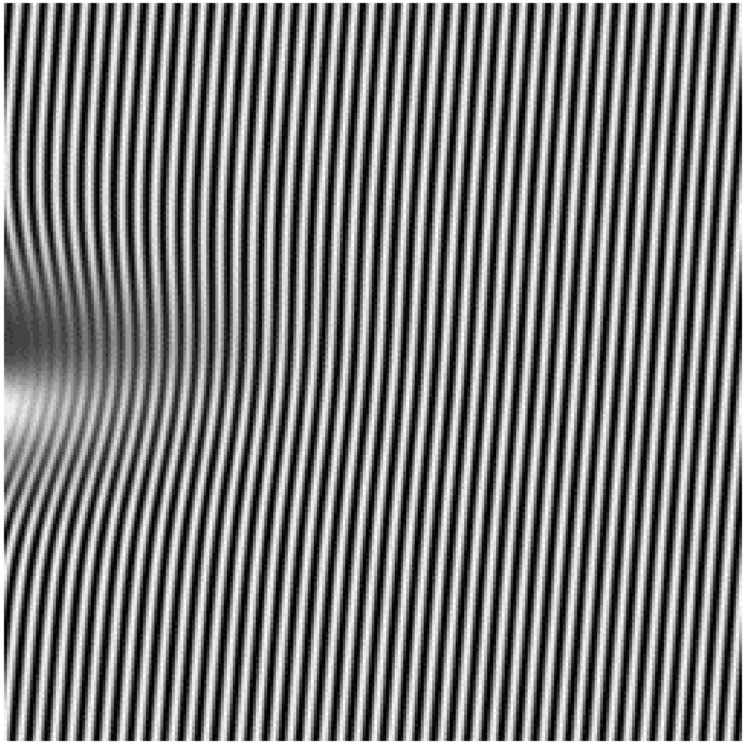
2 Degrees of Freedom



Synthetic Interferogram

Complex

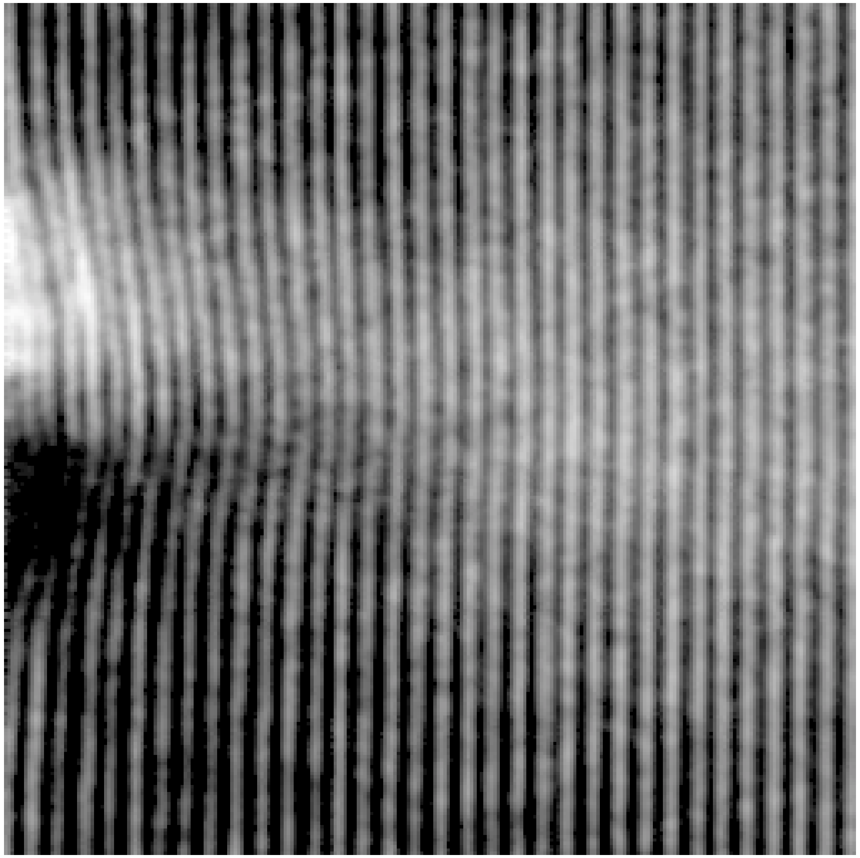
3 Degrees of Freedom



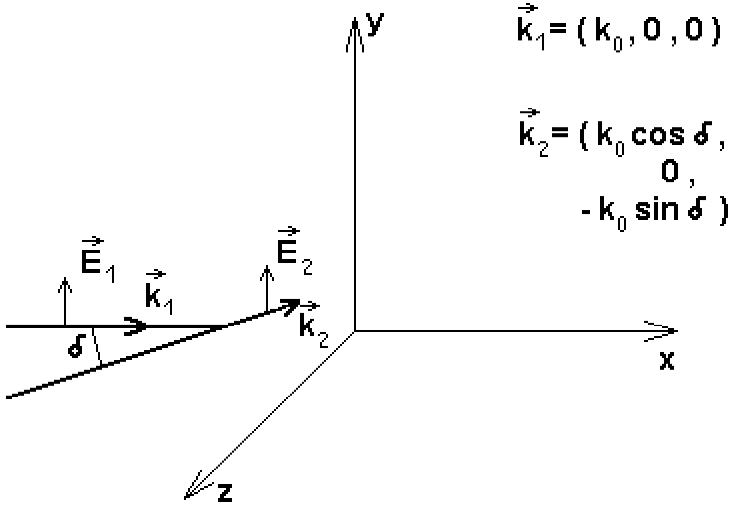
Experimental Interferogram

Complex

3 Degrees of Freedom



2. How is an Interferogram Created



$$\vec{E}_1 = \underline{E(y,z)}\cos[\omega t - \vec{k}_1\vec{r} + \underline{\Phi(y,z)}]\hat{y}$$ probe beam

$$\vec{E}_2 = E_0\cos(\omega t - \vec{k}_2\vec{r})\hat{y}$$ reference beam

$$\Phi(y,z)$$

Probe Beam *Phase Shift*

$$E(y,z)$$

Probe Beam *Amplitude*

$$I(y,z) \propto \left\langle (\vec{E}_1 + \vec{E}_2)^2 \right\rangle_t$$

Intensity at the plane of interference

$$\left\langle (\vec{E}_1 + \vec{E}_2)^2 \right\rangle_t = \left\langle \vec{E}_1{}^2 \right\rangle_t + \left\langle \vec{E}_2{}^2 \right\rangle_t + 2\left\langle \vec{E}_1 \vec{E}_2 \right\rangle_t$$

where

$$\left\langle \vec{E}_1{}^2 \right\rangle_t = \frac{1}{2} E^2(y,z)$$

$$\left\langle \vec{E}_2{}^2 \right\rangle_t = \frac{1}{2} E_0{}^2$$

$$\left\langle \vec{E}_1 \vec{E}_2 \right\rangle_t = \frac{1}{2} E(y,z) E_0 \cos\left[(\vec{k}_2 - \vec{k}_1)\vec{r} + \Phi(y,z) \right]$$

Introducing *new amplitudes*

$$a_0 \propto \frac{E_0}{\sqrt{2}} \qquad \text{and} \qquad a(y,z) \propto \frac{E(y,z)}{\sqrt{2}}$$

we can arrive to the following expression for the interference pattern - *interferogram*

$$i(y,z) = a_0^{\ 2} + a^2(y,z) + \\ +2a_0 a(y,z)\cos\left[2\pi(\omega_0 y + \nu_0 z) + \varphi(y,z)\right]$$

Here ω_0 and ν_0 are *spatial frequences* in the directions *y* and *z* (in the plane of the interferogram) respectively and $\varphi(y,z)$ is the phase shift between the probe beam and the reference beam which has a *constant value* $\varphi_0(y,z) \in (-\pi,\pi>$ *in the signal free region* of the interferogram.

3. Fourier Method of Analysis (Stationary Objects)

Using the formula $\cos x = (e^{ix} + e^{-ix}) / 2$

the expression for an interferogram takes the form

$$i(y,z) = b(y,z) + v(y,z)\exp[2\pi i(\omega_0 y + \upsilon_0 z)] + \\ + v^*(y,z)\exp[-2\pi i(\omega_0 y + \upsilon_0 z)]$$

where

$$b(y,z) = a_0^2 + a^2(y,z)$$ background

$$v(y,z) = a_0 a(y,z)\exp[i\varphi(y,z)]$$ visibility

Provided the *normalized* *visibility* and *background* can be found

$$\bar{v}(y,z) = \frac{v(y,z)}{v_0} = \frac{a_0 a(y,z)\exp[i\varphi(y,z)]}{a_0^2} = \bar{a}(y,z)\exp[i\varphi(y,z)]$$

VISIBILITY (normalized)

$$\bar{b}(y,z) = \frac{b(y,z)}{b_0} = \frac{a_0^2 + a^2(y,z)}{2a_0^2} = \frac{1}{2}[1+\bar{a}^2(y,z)]$$

BACKGROUND (normalized)

(here the quantities ν_0 and b_0 denotes the corresponding values from the *signal free* region) the following quantities can be determined from interferograms of *stationary objects*

$$\varphi(y,z) = \mathrm{Im}[\ln \bar{v}(y,z)]$$

$$\varphi(y,z) = \arctan \frac{\mathrm{Im}[\bar{v}(y,z)]}{\mathrm{Re}[\bar{v}(y,z)]}$$

} **PHASE SHIFT**

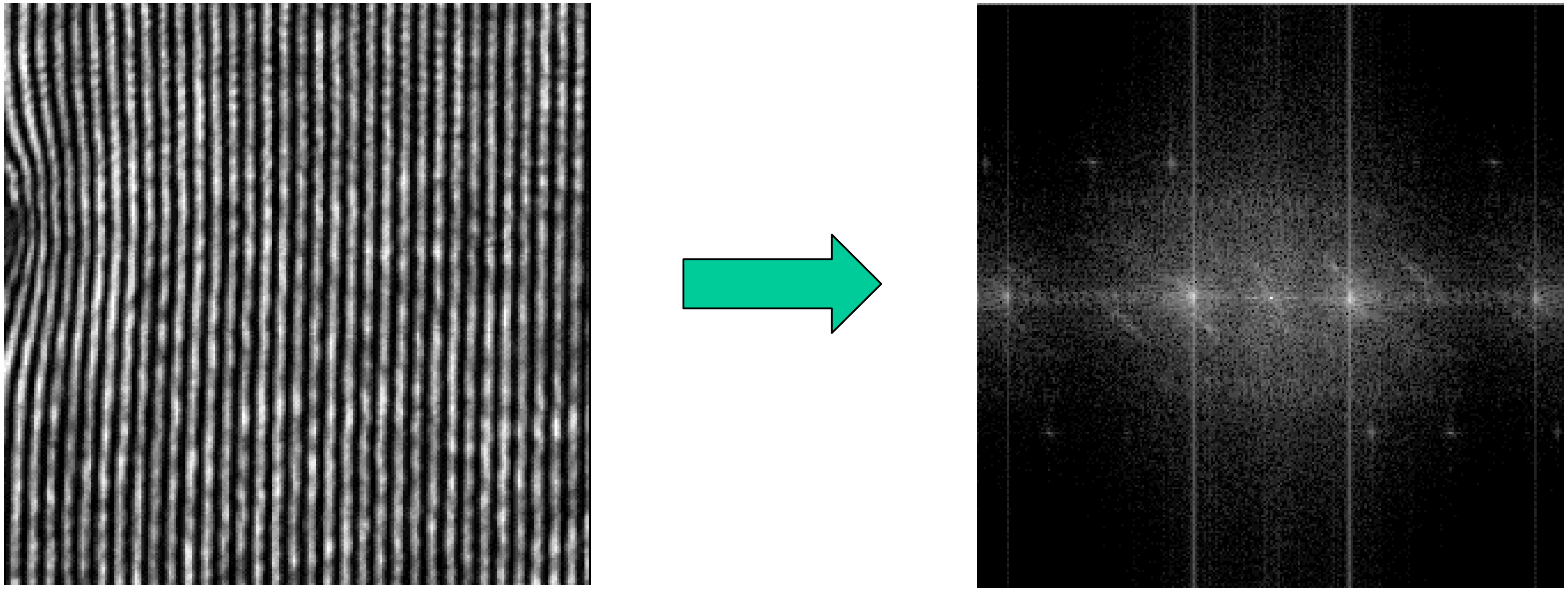
$$\bar{a}(y,z) = \left|\bar{v}(y,z)\right|$$

$$\bar{a}(y,z) = \sqrt{2\bar{b}(y,z) - 1}$$

} **AMPLITUDE**

To get the quantities $\nu(y,z)$ and $b(y,z)$ the *Fourier transform* method can be put to a good use.

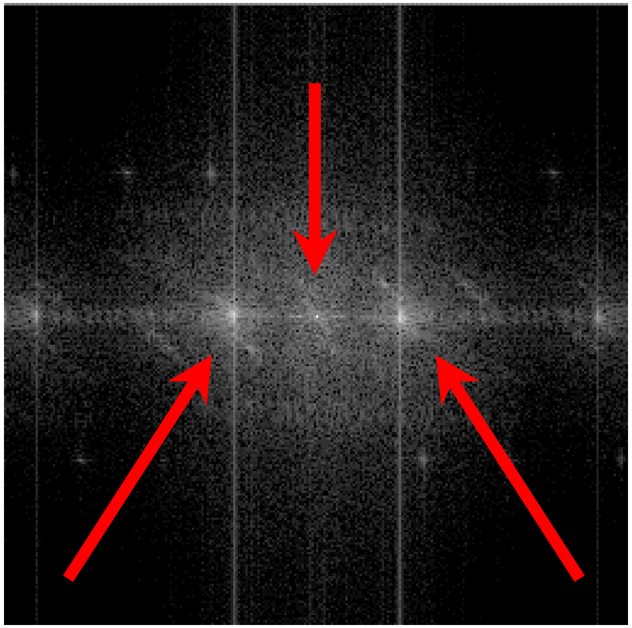
First of all the Fourier transform of the interferogram should be performed with the graphical representation of such process looking like this



Interferogram

Spectrum

Three distinct regions of data – *lobes* - in the *spectral plane* are clearly visible



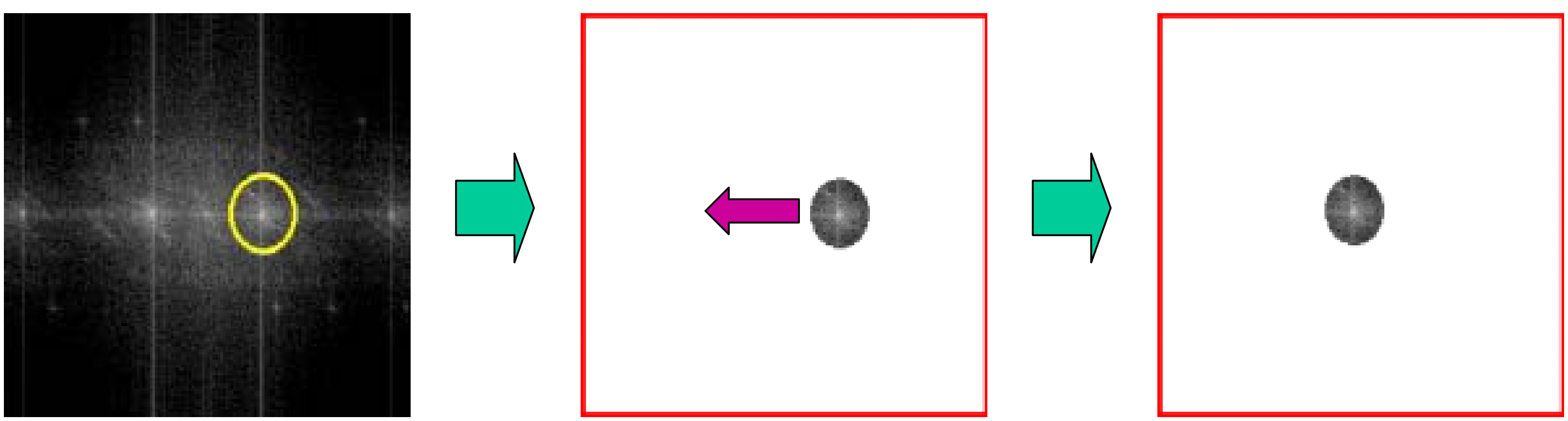
corresponding to the following analytic expression

$$I(\omega,\upsilon)=B(\omega,\upsilon)+V(\omega-\omega_0,\upsilon-\upsilon_0)+V^*(\omega+\omega_0,\upsilon+\upsilon_0)$$

Middle lobe　　　Right lobe　　　Left lobe

The quantity $\nu(y,z)$ can be obtained either from the *right lobe* or the *left lobe* of the spectrum.

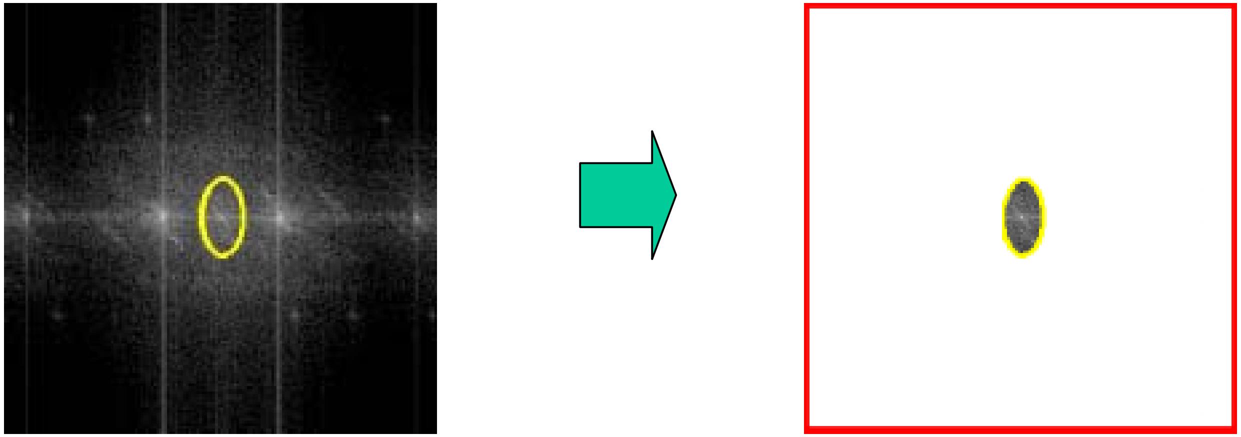
E.g. in the case of the *right lobe* selection (as indicated on the picture bellow) first of all the corresponding part of the spectrum must be identified (yellow ellipse) and the part of the spectrum outside of the elliptical area put to *zero values*. Then the selected elliptical area must be *shifted* to the *central part* of the spectral plane. Finally the *inverse Fourier transform* should be performed to get the $\nu(y,z)$.



To describe this process *mathematically* the following flowchart could be used

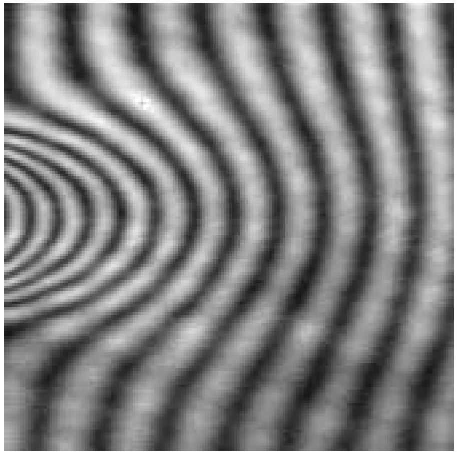
$$V(\omega-\omega_0,\upsilon-\upsilon_0)\rightarrow V(\omega,\upsilon)\rightarrow v(y,z)$$

The quantity *b*(*y,z*) can be obtained from the *middle lobe* *B*(ω,ν) of the spectrum the similar way (no shifting needed in this case).

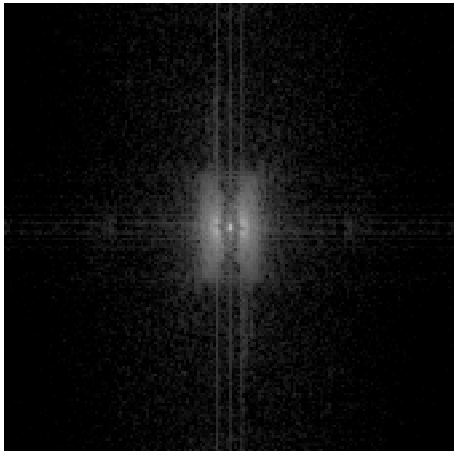


4. Practical Implementation

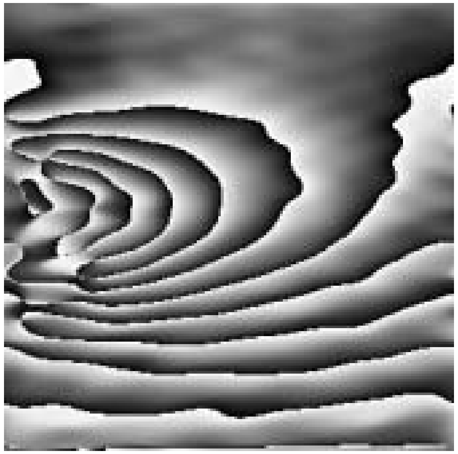
- Shifting of the *side lobe* to the center of the spectral plane can only be done with a certain degree of accuracy. The error in the side lobe shift will cause the reconstructed *phase shift* to be superimposed with an oblique plane. This error can be minimized by the method of *regression by plane* provided some *signal free region* of the interferogram is available.
- Neither the *complex logarithm* nor the *arc tan* functions used as alternatives to reconstruct the phase shift can return values outside the interval $(-\pi,\pi >$. Thus some post processing must be performed to remove artificial jumps in the reconstructed phase shift generated during analysis of interferograms with larges phase shifts.



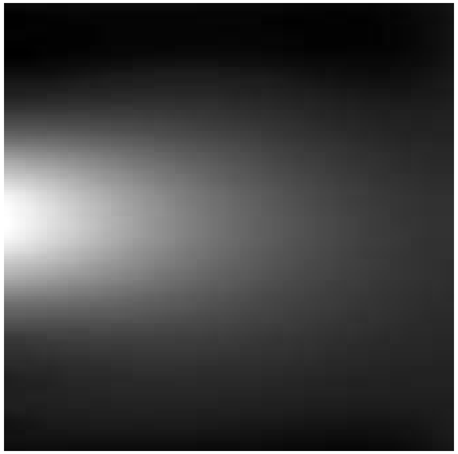
Interferogram



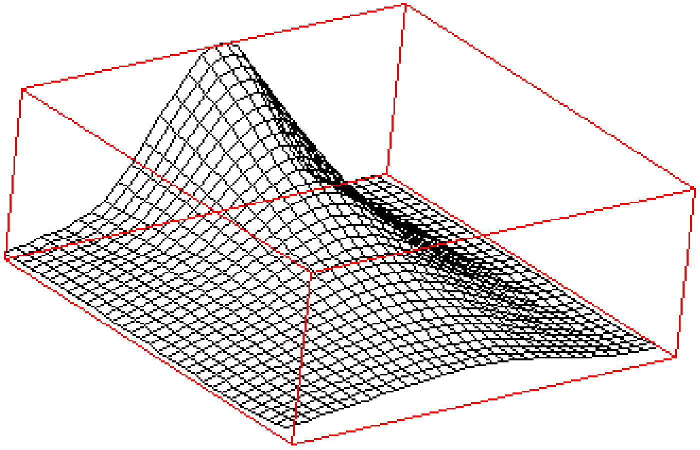
Spectral plane



Phase (with jumps)



Phase (jumps removed)



3D - Phase

- For the best *separation* of side lobes from the middle lobe the high number of fringes is necessary. In an ideal case this number should be close to N/3 (N being the number of digitization points in one row of the interferogram).
- It is important to make sure that the interferogram will contain the *whole object with signal free regions* at all boundaries (with possible exception of one boundary adjacent to the target surface).
- Interferograms without suitable signal free regions can also be analyzed provided an *auxiliary totally signal free interferogram* is available for the exactly the same configuration.

5. Fourier Method of Analysis (Non-Stationary Objects)

When making interferometry of non-stationary objects the *final interferogram* *i*(*y*,*z*) is a *superposition* of a series of *instantaneous interferograms* *i*(*y*,*z*,*t*) taken trough the duration of the probing beam pulse *f*(*t*)

$$i(y,z,t)=a_0^{\,2}f(t)+a^2(y,z,t)f(t)+$$
$$+2a_0a(y,z,t)\cos[2\pi(\omega_0 y+\nu_0 z)+\varphi(y,z,t)]f(t)$$

i.e.

$$i(y,z)=\int_{-\infty}^{+\infty} i(y,z,t)\,dt$$

Let us now suppose that, in principle, both the *phase shift* $\varphi(y,z,t)$ and the *amplitude* $a(y,z,t)$ of the probing beam can evolve in time due to temporal changes of characteristics of the object under investigation.

Keeping this in mind it becomes useful to express these quantities in the form of the *first order Taylor expansion* with static values $\varphi_0(y,z)$ and $a_0(y,z)$ as well as the corresponding time derivatives taken at the *center of gravity of the probe beam pulse*.

$$\varphi(y,z,t) = \varphi_0(y,z) + \varphi_0'(y,z)t$$

$$a(y,z,t) = a_0(y,z) + a_0'(y,z)t$$

The shape of the pulse *f*(*t*) can be chosen (without any lose of generality) to satisfy the following criteria

$$f(t) \geq 0$$

Intensity cannot be *negative*

$$\int_{-\infty}^{+\infty} f(t)\,dt = 1$$

Intensity can be *normalized*

Provided the pulse shape is *symmetric* around its center of gravity, the following useful expression holds

$$f(-t) = f(t) \Rightarrow \int_{-\infty}^{+\infty} t f(t)\,dt = 0$$

After the integration in time has been performed the *background* and *visibility* functions will read as follows

$$\bar{b}_0(y,z)=\frac{1}{2}[1+\bar{a}_0{}^2(y,z)]$$

$$\bar{v}_0(y,z)=\bar{a}_0(y,z)\exp[i\varphi_0(y,z)]\,q(y,z)$$

where

$$q(y,z)=\int_{-\infty}^{+\infty}\cos[\varphi'_0(y,z)t]f(t)dt$$

is a new *modifying function* $0<q(y,z)\leq 1$

Knowing the pulse shape (either numerically or analytically) the phase shift time derivative $\varphi_0'(y,z)$ can be determined from the modifying function $q(y,z)$.

E.g. for the Gaussian pulse

$$f(t) = \frac{1}{\sqrt{\pi}\tau} \exp\left(-\frac{t^2}{\tau^2}\right)$$

we get

$$\left|\varphi_0'(y,z)\right| = \frac{2}{\tau}\sqrt{-\ln q(y,z)}$$

It is also becoming clear that in the case of non-stationary objects it is not possible to directly use the normalized visibility to calculate the normalized amplitude as now the absolute value of the normalized visibility is equal to the *product* of the normalized amplitude and the modifying function

$$\left|\bar{v}_0(y,z)\right| = \bar{a}_0(y,z)\, q(y,z)$$

The only option left for determining the normalized amplitude in this case is using the normalized background. The absolute value of the normalized visibility is then used to determine the modifying function.

QUANTITIES WHICH CAN BE RECONSTRUCTED FROM *COMPLEX INTERFEROGRAMS*

$$\varphi_0(y,z) = \mathrm{Im}[\ln \bar{v}_0(y,z)]$$

PHASE SHIFT

$$\bar{a}_0(y,z) = \sqrt{2\bar{b}_0(y,z) - 1}$$

AMPLITUDE

$$q(y,z) = \frac{\left|\bar{v}_0(y,z)\right|}{\sqrt{2\bar{b}_0(y,z) - 1}}$$

Q-FUNCTION

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