Construction of doubly periodic solutions via the Poincare–Lindstedt method in the case of massless φ^4 theory.

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Abstract

Doubly periodic (periodic both in time and in space) solutions for the Lagrange–Euler equation of the $(1+1)$–dimensional scalar φ^4 theory are studied. Provided that nonlinear term is small, the Poincare–Lindstedt asymptotic method is used to find asymptotic solutions in the standing wave form. It is proved that using the Jacobi elliptic function **cn** as the zero approximation one can solve the problem of the main resonance, appearing in the case of zero mass, and construct a uniform expansion.

1 Introduction

Our investigation is dedicated to the construction of doubly periodic classical fields in the $(1+1)$–dimensional φ^4 theory. We study the model of an isolated real scalar field $\varphi(x,t)$, described by the Lagrangian density:

$$\mathcal{L}(\varphi) = \frac{1}{2}\left(\varphi_{,t}^2(x,t) - \varphi_{,x}^2(x,t) - M^2\varphi^2(x,t) - \frac{\varepsilon}{2}\varphi^4(x,t)\right).$$

Let us consider the Lagrange–Euler equation:

$$\frac{\partial^2\varphi(x,t)}{\partial x^2} - \frac{\partial^2\varphi(x,t)}{\partial t^2} - M^2\varphi(x,t) - \varepsilon\varphi^3(x,t) = 0. \tag{1}$$

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There are two classes of doubly periodic solutions for this equation. If we seek fields in the traveling wave form:

$$\varphi(x,t) = \varphi(x - vt),$$

where v is the velocity of the wave motion, then equation (1) reduces to the Duffing's equation [1]. As is known [2], periodic solutions for this equation are Jacobi elliptic functions. Such solutions are well known for both $(1+1)$–dimensional [3] and $(3+1)$–dimensional [4] models of φ^4 theory.

The second class of doubly periodic solutions consists of functions in the standing wave form:

$$\varphi(x,t) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \mathcal{C}_{nj} \sin(n(x-x_0)) \sin(j\cdot\omega(t-t_0)), \qquad (2)$$

where x_0 and t_0 are constants determined by boundary and initial conditions. Equation (1) is a translation-invariant one, so, we can restrict our consideration to the case of zero x_0 and t_0 without loss of generality. We suppose that the function $\varphi(x,t)$ is 2π-periodic in space and seek its period in time.

Exact standing wave solutions are not known. Approximate solutions for equation (1) with $\varepsilon = 1$ and $M = 1$ have been found under the assumption that all highest harmonics are zeros: $\forall n, j > N : \mathcal{C}_{nj} = 0$, where N is a large number. Numerical calculations have been made [5] for several values of N. The obtained solutions are very close to each other, but convergence of the sequence of these solutions as N tends to infinity has yet to be proved.

Provided that $\varepsilon \ll 1$, C. Eugene Wayne [6] has considered the problem of the construction of periodic and quasi-periodic in time solutions for the equation (1) with the mass term, depending on space-coordinate x. He has defined non-resonance conditions for eigenvalues of the operator $L_M \equiv \frac{\partial^2}{\partial x^2} - M^2(x)$ and proved that if these conditions are satisfied, then one can construct periodic and quasi-periodic solutions for equation (1), using a variant of the Kolmogorov, Arnold, Moser (KAM) scheme. C. Eugene Wayne has remarked that although his theorem gives one many periodic solutions, one has no information whether or not they occur in smooth families.

2 Asymptotic solutions in the standing wave form

Let us consider periodic solutions for equation (1), provided that ε is small or, equivalently, that these solutions have small amplitudes.

An asymptotic expansion, containing only bounded functions, is called *a uniform expansion.* The possibility to obtain a uniform expansion, using standard asymptotic methods [7], for example, Poincare–Lindstedt [8] or Krylov–Bogoliubov [9] ones, depends on values of the frequencies in zero approximation. If $\varepsilon = 0$, then equation (1) is a linear one and has periodic solutions of the form (2) with frequencies in time $\Omega_j = \sqrt{j^2 + M^2}$, where $j \in \mathbb{N}$. There are two fundamentally different cases.

If

$$\forall i,j \in \mathbb{N} \qquad \nexists\, k,n \in \mathbb{N} \quad \text{such that} \quad \frac{\Omega_i}{\Omega_j} = \frac{k}{n} \ ,$$

then it is a non-resonance case.

The periodic asymptotic solutions in non-resonance case are well known. To obtain these solutions with any degree of exactness standard asymptotic methods can be used.

The resonance case, when there exist two frequencies Ω_j, whose relation is rational, is more difficult. The Krylov–Bogoliubov method and the standard variant of the Poincare–Lindstedt method allow to find periodic solutions only in a few leading orders in ε.

The important example of resonance case is the massless φ^4 theory when $\Omega_j = j$ and all relations of frequencies are rational. It is the main resonance case. Using the standard asymptotic methods one cannot construct a periodic solution even to the first order in ε. It is possible to transform differential equations in a system of nonlinear algebraic equations in Fourier coefficients and frequencies using the Poincare–Dulac's normal form method. The algorithm of this procedure was constructed [10, 11] and realized in the software for symbolic and algebraic computation **REDUCE** [12] (the system **NORT** [13, 14]). But algorithm to solve the obtained algebraic system has yet to be realized.

3 The standing wave solutions for the massless φ^4 theory

3.1 Construction of asymptotic solutions via the Poincare–Lindstedt method

The purpose of this article is the construction of standing wave solutions of equation (1) with $M = 0$, using asymptotic methods. They can only be applied provided $\varepsilon \ll 1$.

We use the Poincare–Lindstedt method: introduce the new time $\tilde{t} \equiv \omega t$ and seek a doubly periodic solution of equation (1) $\varphi(x, \tilde{t})$ and the frequency (in time) ω in the form of power series in ε:

$$\varphi(x, \tilde{t}, \varepsilon) \equiv \sum_{n=0}^{\infty} \varphi_n(x, \tilde{t})\varepsilon^n,$$
$$\omega(\varepsilon) \equiv 1 + \sum_{n=1}^{\infty} \omega_n \varepsilon^n.$$

After we expand the Lagrange–Euler equation in a power series in ε and obtain a sequence of equations. Write two leading equations:
1) To zero order in ε the equation in φ_0 is:

$$\frac{\partial^2 \varphi_0(x, \tilde{t})}{\partial x^2} - \frac{\partial^2 \varphi_0(x, \tilde{t})}{\partial \tilde{t}^2} = 0, \tag{3}$$

2) To first order in ε the equation in φ_0, φ_1 and ω_1 is:

$$\frac{\partial^2 \varphi_1(x, \tilde{t})}{\partial x^2} - \frac{\partial^2 \varphi_1(x, \tilde{t})}{\partial \tilde{t}^2} = 2\omega_1 \frac{\partial^2 \varphi_0(x, \tilde{t})}{\partial \tilde{t}^2} + \varphi_0^3(x, \tilde{t}). \tag{4}$$

Equation (3) has many periodic solutions. If we select as a solution for this equation the function $\varphi_0(x, \tilde{t}) = \sin(x)\sin(\tilde{t})$, then the second equation hasn't periodic solutions, because the frequency of the external force $\sin(3x)\sin(3\tilde{t})$ is equal to the frequency of its own oscillations and it is impossible to put this resonance harmonic to zero, selecting only ω_1. In massless case correct selection of not only the frequency $\omega(\varepsilon)$, but also the function $\varphi_0(x,\tilde{t})$ allows to find a uniform expansion.

3.2 The condition of existence of periodic solutions

The general solution in the standing wave form (2) for equation (3) is the function

$$\varphi_0(x,\tilde{t}) = \sum_{n=1}^{\infty} a_n \sin(nx)\sin(n\tilde{t})$$

with arbitrary a_n. We have to find such coefficients a_n that the function $\varphi_1(x,\tilde{t})$ is a periodic solution for equation (4). If we select $\varphi_1(x,\tilde{t})$ as a double sum:

$$\varphi_1(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} b_{nj}\sin(nx)\sin(j\tilde{t})$$

with arbitrary b_{nj}, then equation (4) can be presented in the form of Fourier series:

$$\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} - 2\omega_1\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} - \varphi_0^3(x,\tilde{t}) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} R_{nj}(\mathbf{a},\mathbf{b})\sin(nx)\sin(j\tilde{t}) = 0.$$

and is equivalent to the following infinite system of the algebraic equations in Fourier coefficients of the functions $\varphi_0(x,\tilde{t})$ and $\varphi_1(x,\tilde{t})$:

$$\forall n,j \ : \ R_{nj}(\mathbf{a},\mathbf{b}) = 0. \tag{4$'$}$$

This system has a subsystem of the equations in Fourier coefficients of the function $\varphi_0(x,\tilde{t})$:

$$\forall j\in\mathbb{N} \ : \ R_{jj}(\mathbf{a}) = 0, \tag{5}$$

where

$$\begin{aligned} R_{jj} \ \equiv\ & 9a_j^3 + 3a_j^2a_{3j} + a_j\left(6\sum_{\substack{s=1\\ s\neq j}}^{\infty}(2a_s^2 + a_sa_{2j+s}) + 3\sum_{\substack{s=1\\ s\neq j}}^{2j-1} a_sa_{2j-s} - 32j^2\omega_1\right) + \\ &+\ 3\sum_{\substack{s=1\\ s\neq j}}^{\infty}\sum_{\substack{p=1\\ p\neq j}}^{\infty} a_sa_pa_{j+s+p} + 3\sum_{\substack{s=1\\ s\neq j}}^{\infty}\sum_{\substack{p=1\\ p\neq j\\ p\neq 2j-s}}^{\infty} a_sa_pa_{s+p-j} + \sum_{s=1}^{j-2}\sum_{p=1}^{j-2} a_sa_pa_{j-s-p}\end{aligned}$$

We have obtained the necessary and sufficient condition of the existence of periodic solutions for equation (4): there exists a periodic function $\varphi_1(x,\tilde{t})$, satisfying equation (4), if and only if the Fourier coefficients of the function $\varphi_0(x,\tilde{t})$ satisfy system (5).

Coefficient a_1 is a parameter determining the oscillation amplitude. In fact, let $a_j = c_ja_1$ and $\omega_1 = c_\omega a_1^2$, then all polynomials $R_{jj}(\mathbf{a})$ are proportional to a_1^3: $R_{jj}(\mathbf{a}) = a_1^3R_{jj}(\mathbf{c})$ and, therefore, the coefficient a_1 can be selected arbitrarily.

3.3 The approximate solution of system (5)

This system of equations is very difficult to solve. On the one hand, all R_{jj} are infinite series and the number of equations is infinite too. On the other hand, each equation of this system is a nonlinear one. We have restricted ourselves to finding a particular solution. To simplify calculations we assume that the function $\varphi_0(x,\tilde{t})$ contains only odd harmonics. Our goal is to find a real solution and we seek $c_j \in \mathbb{R}$.

To find an approximate solution we apply the Galerkin method: cut off higher diagonal harmonics $\forall j > N : c_{2j-1} = 0$ and seek an approximation for $\varphi_0(x,\tilde{t})$ in the following form

$$\varphi_0(x,\tilde{t}) = a_1 \left\{ \sum_{j=1}^{N} c_{2j-1} \sin((2j-1)x) \sin((2j-1)\tilde{t}) \right\}.$$

We have obtained the finite system of nonlinear equations, with the number of equations three times as mush as that of the variables. We solve the first N equations and substitute the obtained values in the other equations. The system of the N leading nonlinear equations can be solved, using standard procedures of the computer algebra system **REDUCE**: the **SOLVE** operator and the Groebner basis package [15, 16]. The **SOLVE** operator (in "on rounded" mode) round off numbers with accuracy $\delta = 10^{-11}$ (the system (5) is too difficult to be solved in "off rounded" mode). We admit that the system of the first N equations is solved if for all $j \leq N$ the inequalities $|R_{jj}(\mathbf{c})| < \delta$ are true. We also admit that the value of N is sufficient to solve system (5) if for all $j \in \mathbb{N}$ the inequalities $|R_{jj}(\mathbf{c})| < \delta$ are true.

Using the **SOLVE** operator and computer with 128M bytes operating memory (RAM) we have solved the system (5) only for $N < 5$, but due to a short program (see Appendix) we have found a particular solution for any $N < 50$ and obtained that the minimal sufficient value of N for $\delta = 10^{-11}$ is $N = 8$ and that the value of the frequency is

$$\omega_1 = 0.282680034541 a_1^2.$$

We also have found numerical values of the fifteen leading Fourier coefficients of $\varphi_0(x,\tilde{t})$. The obtained values of c_j are very close to the values of the corresponding terms of the following finite sequence:

$$\mathbf{d} = \{d_{2j-1} = \frac{f_{2j-1}}{f_1}, \ \text{where} \quad f_{2j-1} \equiv \frac{q^{j-1/2}}{1+q^{2j-1}}; \ d_{2j} = 0; \ j < 23\},$$

with $q = 0.0142142623201$. It is easy to verify that substitution of this finite sequence gives

$$\forall j \in \mathbb{N} \quad : \quad |R_{jj}(\mathbf{d})| < 10^{-12}.$$

In other words, the finite sequence $\mathbf{d}$ is an approximate solution of system (5). These numerical calculations help to find the analytical form of $\varphi_0(x,\tilde{t})$. The following table illustrates this interesting result:

j	c_j	$R_{jj}(\mathbf{c})$	d_j	$R_{jj}(\mathbf{d})$
1	1	0	1	0
3	$1.44162661711 \times 10^{-2}$	3.5×10^{-12}	$1.44162661711 \times 10^{-2}$	-8.0×10^{-14}
5	$2.04917177408 \times 10^{-4}$	2.1×10^{-12}	$2.04917177419 \times 10^{-4}$	-3.4×10^{-15}
7	$2.91274649724 \times 10^{-6}$	7.8×10^{-12}	$2.91274651543 \times 10^{-6}$	-9.7×10^{-17}
9	$4.14025418115 \times 10^{-8}$	8.3×10^{-13}	$4.14025430425 \times 10^{-8}$	-2.3×10^{-18}
11	$5.88506592014 \times 10^{-10}$	1.0×10^{-14}	$5.88506607528 \times 10^{-10}$	-4.9×10^{-20}
13	$8.36488192079 \times 10^{-12}$	4.6×10^{-13}	$8.36518729655 \times 10^{-12}$	-9.7×10^{-22}
15	$1.18901919266 \times 10^{-13}$	-7.8×10^{-22}	$1.1890496659 \times 10^{-13}$	-1.8×10^{-23}
17	0	4.4×10^{-12}	$1.69014638629 \times 10^{-15}$	-3.4×10^{-25}
19	0	7.3×10^{-14}	$2.40241840942 \times 10^{-17}$	-6.0×10^{-27}
21	0	1.1×10^{-15}	$3.41486054743 \times 10^{-19}$	-1.0×10^{-28}
23	0	1.7×10^{-17}	$4.85397236079 \times 10^{-21}$	-1.8×10^{-30}
25	0	2.4×10^{-19}	$6.89956364312 \times 10^{-23}$	-3.0×10^{-32}
27	0	3.3×10^{-21}	$9.8072207518 \times 10^{-25}$	-4.9×10^{-34}
29	0	4.2×10^{-23}	$1.39402408398 \times 10^{-26}$	-8.1×10^{-36}
31	0	5.2×10^{-25}	$1.98150240103 \times 10^{-28}$	-1.3×10^{-37}
33	0	5.7×10^{-27}	$2.81655949163 \times 10^{-30}$	-2.1×10^{-39}
35	0	6.1×10^{-29}	$4.00353154544 \times 10^{-32}$	-3.4×10^{-41}
37	0	6.2×10^{-31}	$5.69072475939 \times 10^{-34}$	-5.4×10^{-43}
39	0	5.9×10^{-33}	$8.08894545219 \times 10^{-36}$	-8.5×10^{-45}
41	0	5.0×10^{-35}	$1.14978392551 \times 10^{-37}$	-1.3×10^{-46}
43	0	3.6×10^{-37}	$1.63433303287 \times 10^{-39}$	-2.1×10^{-48}
45	0	1.7×10^{-39}	$2.32308384477 \times 10^{-41}$	-1.6×10^{-49}
47	0	0	0	1.4×10^{-50}

3.4 The exact solution of system (5)

The aim of this section is to find the analytical form of the function, which Fourier coefficients satisfy system (5). For arbitrary $q \in (0,1)$ let us define the following sequence:

$$\mathbf{f} \stackrel{\text{def}}{=} \{\ \forall n \in \mathbb{N} : f_{2n-1} = \frac{q^{n-1/2}}{1+q^{2n-1}}, f_{2n} = 0\ \}.$$

The terms of the sequence $\mathbf{f}$ are proportional to Fourier coefficients of the Jacobi elliptic function **cn** [2]:

$$\mathrm{cn}(z,k) = \frac{\gamma}{k}\sum_{n=1}^{\infty} f_{2n-1}\cos\left((2n-1)\frac{\gamma z}{4}\right), \quad \text{where} \quad \gamma \equiv \frac{2\pi}{K}\ , \quad z \in \mathbb{R}. \tag{6}$$

Let us clarify introduced notations and point out some properties of the elliptic cosine:
1) Basic periods of the doubly periodic function $\mathrm{cn}(z,k)$ are $4K(k)$ and $2K(k)+2iK'(k)$, where $K(k)$ is a full elliptic integral, $K'(k) \equiv K(k')$ and $k' = \sqrt{1-k^2}$.
2) The parameter q in the Fourier expansion can be expressed in terms of elliptic integrals: $q \equiv e^{-\pi\frac{K'}{K}}$.

3) The Fourier-series expansion of the function $\mathrm{cn}(z,k)$ does not include even harmonics. This expansion is valid in the following domain of the complex plane: $-K' < \Im m\, z < K'$, in particular, for $z \in \mathbb{R}$.

4) If $z \in \mathbb{R}$ and $k \in (0,1)$, then $\mathrm{cn}(z,k) \in \mathbb{R}$.

5) The function $\mathrm{cn}(z,k)$ is a solution of the following differential equation:

$$\frac{d^2\,\mathrm{cn}(z,k)}{dz^2} = (2k^2-1)\,\mathrm{cn}(z,k) - 2k^2\,\mathrm{cn}^3(z,k). \tag{7}$$

The latest property means that the infinite sequence of the Fourier coefficients for $\mathrm{cn}(z,k)$ is a solution of some infinite system of nonlinear algebraic equations. Let us find this system.

On the one hand, it is clear from (6) that the Fourier-series expansion for the function $\mathrm{cn}^3(z,k)$ is:

$$\mathrm{cn}^3(z,k) = \frac{\gamma^3}{4k^3}\sum_{j=1}^{\infty} F_j^{(3)}(\mathbf{f})\cos\left(j\frac{\gamma z}{4}\right), \quad \text{where } j = 1,3,5,\ldots,+\infty\ ;$$

$$\begin{aligned} F_j^{(3)}(\mathbf{f}) \;=\; & 3f_j^3 + 3f_j^2 f_{3j} + f_j\left(6\sum_{\substack{s=1\\ s\neq j}}^{\infty}(f_s^2 + f_s f_{2j+s}) + 3\sum_{\substack{s=1\\ s\neq j}}^{2j-1} f_s f_{2j-s}\right) + \\ & + 3\sum_{\substack{s=1\\ s\neq j}}^{\infty}\sum_{\substack{p=1\\ p\neq j}}^{\infty} f_s f_p f_{j+s+p} + 3\sum_{\substack{s=1\\ s\neq j}}^{\infty}\sum_{\substack{p=1\\ p\neq j\\ p\neq 2j-s}}^{\infty} f_s f_p f_{s+p-j} + \sum_{s=1}^{j-2}\sum_{p=1}^{j-2} f_s f_p f_{j-s-p} \end{aligned}$$

(in all sums we summarize over only odd numbers).

On the other hand, from the differential equation (7) it follows that $F_j^{(3)}(\mathbf{f})$ is proportional to f_j, with coefficients of proportionality depending on j:

$$\forall j \;:\; F_j^{(3)}(\mathbf{f}) = \left(\frac{2(2k^2-1)}{\gamma^2} + \frac{j^2}{8}\right) f_j. \tag{8}$$

Thus, the sequence $\mathbf{f}$ is a nonzero solution of system (8) at all $q \in (0,1)$. The following lemma proves the existence of a preferred value of q.

Lemma. *There exists such value of parameter $q \in (0,1)$ that the sequence $\mathbf{f}$ is a real solution of system (5), in addition a value of ω_1 also is real.*

Proof. Inserting the sequence $\mathbf{f}$ into system (5) : $a_j = f_j$ and using system (8), we obtain:

$$\begin{aligned} \forall j \;:\; R_{jj}(\mathbf{f}) &= \left\{F_j^{(3)}(\mathbf{f}) + f_j\left(6\sum_{n=1}^{\infty} f_n^2 - 32j^2\omega_1\right)\right\} = \\ &= f_j\left\{6\sum_{n=1}^{\infty} f_n^2 + \frac{2(2k^2-1)}{\gamma^2} + j^2\left(\frac{1}{8} - 32\omega_1\right)\right\} = 0. \end{aligned}$$

System (5) has a nonzero solution if and only if

$$\begin{cases} \omega_1 = \dfrac{1}{256}, \\ \sum\limits_{n=1}^{\infty} f_n^2 = \dfrac{(1-2k^2)}{3\gamma^2}. \end{cases}$$

We have obtained the value of ω_1. The second equation of this system is equivalent to the following equation in parameter q:

$$3\sum_{n=1}^{\infty}\left(\frac{q^{n-1/2}}{1+q^{2n-1}}\right)^2 - \left(\frac{1}{4}+\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\right)^2 + 2\left(\sum_{n=1}^{\infty}\frac{q^{n-1/2}}{1+q^{2n-1}}\right)^2 = 0. \qquad (9)$$

This equation has the following solution on interval (0,1):

$$q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13}.$$

Thus the lemma is proved.

Now it is easy to construct the required zero approximation for the function $\varphi(x,\tilde{t})$:

$$\varphi_0(x,\tilde{t}) = A\Big\{\mathrm{cn}(\alpha(x-\tilde{t}),k) - \mathrm{cn}(\alpha(x+\tilde{t}),k)\Big\}.$$

For arbitrary $k \in (0,1)$ this function is a real solution of equation (4). If $\alpha = \frac{2K}{\pi}$, then the periods of $\varphi_0(x,\tilde{t})$ in x and in $\tilde{t}$ are equal to 2π. Using the Fourier-series expansion for the function $\mathrm{cn}(z,k)$ (formula (6)), we obtain the following expansion for function $\varphi_0(x,\tilde{t})$:

$$\varphi_0(x,\tilde{t}) = 2A\frac{\gamma}{k}\sum_{n=1}^{\infty} f_{2n-1}\sin((2n-1)x)\sin((2n-1)\tilde{t}).$$

If $q = 1.42142623201 \times 10^{-2} \pm 1 \times 10^{-13}$, then q is a solution of equation (9) and the sequence $\mathbf{f}$ is a real solution of system (5). The middle value of q corresponds to $k = 0.451075598811$, $\gamma = 3.78191440007$ and $\alpha = 1.0576653982$. All equation in system (5) are homogeneous ones, hence, for these values of parameters, the sequence of the Fourier coefficients of the function $\varphi_0(x,\tilde{t})$ also is a solution of system (5), with

$$\omega_1 = \frac{\gamma^2}{64k^2}A^2 = 1.0983600974A^2.$$

Thus we have proved that the function

$$\varphi_0(x,\tilde{t}) = A\Big\{\mathrm{cn}(\alpha(x-\tilde{t}),k) - \mathrm{cn}(\alpha(x+\tilde{t}),k)\Big\},$$

with $k = 0.451075598811$ and $\alpha = 1.0576653982$ is such a standing wave solution of equation (3) that equation (4) has a periodic solution.

3.5 The first approximation

Now it is easy to find this periodic solution $\varphi_1(x,\tilde{t})$. Let us designate the Fourier coefficients of the function $\varphi_0^3(x,\tilde{t})$ as D_{nj}:

$$\varphi_0^3(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} D_{nj}\sin(nx)\sin(j\tilde{t}).$$

Equation (4) gives the following result(b_{nn} are arbitrary numbers):

$$\varphi_1(x,\tilde{t}) = \sum_{n=1}^{\infty}\sum_{\substack{j=1\\ j\neq n}}^{\infty} \frac{D_{nj}}{j^2-n^2}\sin(nx)\sin(j\tilde{t}) + \sum_{n=1}^{\infty} b_{nn}\sin(nx)\sin(n\tilde{t}).$$

It should be noted that the function $\varphi_1(x,\tilde{t})$ with arbitrary diagonal coefficients b_{nn} is a solution of equation (4) and that all off-diagonal coefficients of $\varphi_1(x,\tilde{t})$ are proportional to A^3.

3.6 The second approximation

Let us consider the equation to second order in ε:

$$\frac{\partial^2\varphi_2(x,\tilde{t})}{\partial x^2} - \frac{\partial^2\varphi_2(x,\tilde{t})}{\partial\tilde{t}^2} = 2\omega_1\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} + (2\omega_2+\omega_1^2)\frac{\partial^2\varphi_0(x,\tilde{t})}{\partial\tilde{t}^2} + 3\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t}). \quad (10)$$

If all diagonal coefficients of $\varphi_1(x,\tilde{t})$ are zeros: $\forall n : b_{nn}=0$, then $\forall j,n : b_{jn}=-b_{nj}$, and the function $\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t})$ has no diagonal harmonics. Hence, selecting $\omega_2=-\frac{1}{2}\omega_1^2$, we obtain a periodic solution to equation (10):

$$\varphi_2(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{\substack{j=1\\ j\neq n}}^{\infty} \frac{H_{nj}}{j^2-n^2}\sin(nx)\sin(j\tilde{t}) + \sum_{n=1}^{\infty} h_{nn}\sin(nx)\sin(n\tilde{t}), \quad \text{where}$$

$$H(x,\tilde{t}) \equiv 2\omega_1\frac{\partial^2\varphi_1(x,\tilde{t})}{\partial\tilde{t}^2} - 3\varphi_1(x,\tilde{t})\varphi_0^2(x,\tilde{t}) \equiv \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} H_{nj}\sin(nx)\sin(j\tilde{t}).$$

It should be noted that all diagonal coefficients h_{nn} are arbitrary numbers.

4 Conclusions

Using massless φ^4 theory as an example, we have shown that a uniform expansion of solutions for quasilinear Klein–Gordon equations can be constructed even in the main resonance case. To construct the uniform expansion we have used the Poincare–Lindstedt method and the nontrivial zero approximation: the function

$$\varphi_0(x,t) = A\Big\{\mathrm{cn}(\alpha(x-\omega t),k) - \mathrm{cn}(\alpha(x+\omega t),k)\Big\},$$

with $k = 0.451075598811$ and $\alpha = 1.0576653982$.

Thus, using the Jacobi elliptic function **cn** instead of the trigonometric function **cos**, we have put the main resonance to zero and constructed with accuracy $\mathcal{O}(\varepsilon^3)$ the doubly periodic solution in the standing wave form:

$$\varphi(x,\omega t) = \varphi_0(x,\omega t) + \varepsilon\varphi_1(x,\omega t) + \varepsilon^2\varphi_2(x,\omega t) + \mathcal{O}(\varepsilon^3), \text{ with the frequency}$$

$$\omega = 1+\frac{\gamma^2}{64k^2}A^2\varepsilon - \frac{\gamma^4}{8192k^4}A^4\varepsilon^2 + \mathcal{O}(\varepsilon^3) = 1+1.0983600974A^2\varepsilon - 0.6031974518A^4\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

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Appendix

The procedure "fourier" expands the equation in the Fourier series.

```
procedure fourier(FF,X);
% This procedure constructs Fourier series for polynomials
% of sin(x) and cos(x).
  begin
  scalar F;
  for all AA , BB such that
                 df(AA,X) NEQ 0 AND df(BB,X) NEQ 0
                  let
     cos(AA)*cos(BB)=(cos(AA-BB)+cos(AA+BB))/2,
     sin(AA)*sin(BB)=(cos(AA-BB)-cos(AA+BB))/2,
     sin(AA)*cos(BB)=(sin(AA-BB)+sin(AA+BB))/2,
                      sin(AA)**2=(1-cos(2*AA))/2,
                      cos(AA)**2=(1+cos(2*AA))/2;
          F:=FF;
  for all AA , BB such that
                 df(AA,X) NEQ 0 AND df(BB,X) NEQ 0
                  clear
                       cos(AA)*cos(BB),
               sin(AA)*sin(BB),
                       sin(AA)*cos(BB),
       sin(AA)**2,
       cos(AA)**2;
          return F;
    end;
end;
```

This program have been written in **REDUCE 3.6** and tested on computer with 16M bytes operating memory. The program constructs the system (5): $R_{jj} = 0$ and finds a particular solution of this system. It use the procedure "fourier" to expand the equation in Fourier series. You can change the number of unknowns (parameter n). The number of equation is $3n$. We assume that the solution is real and includes only odd harmonics.

```
out "Rjj.res";
in fourier$
n:=10$
operator a,c,u$
depend u,xx,tt$

% Expand the unknown function u and the equation in Fourier series.

u(xx,tt):=for j:=1:n sum a(2j-1)*sin((2j-1)*xx)*sin((2*j-1)*tt)$
equ1:=2*w1*df(u(xx,tt),tt,2)+fourier(fourier(u(xx,tt)**3,xx),tt)$
equat:={}$

%  The frequency w_1 is proportional to a(1)^2 (c(1)=c_{\omega}).

w1:=c(1)*a(1)**2$
for j:=2:n do  a(2*j-1):=c(2*j-1)*a(1)$
equ1:=equ1$

% Construct the system (5): R_{jj}=0.

for k:=1:3*n do
    equat:=append(equat,
                  {16*df(equ1,sin((2*k-1)*xx),sin((2*k-1)*tt))/a(1)**3});

% Find a real solution of this system.

on rounded$
c(1):=0$
for k:=2:n do
<< for j:=k:n do c(2*j-1):=0;
   clear c(1),c(2*k-1);
   equat1:=first(equat);
   c(1):=c(1)-equat1/df(equat1,c(1));
   equatpr:=part(equat,k);
   respr:=solve(equatpr,c(2*k-1));
   while respr neq {} do
 << c(2*k-1):=part(respr,1,2);
    if(c(2*k-1)=sub(i=-i,c(2*k-1)) and c(2*k-1)<1)
```

```
      then respr:={}
      else respr:=rest(respr);
 >>;
  test:=0;
  while(test=0) do
 << test:=1;
    clear c(1);
    equat1:=first(equat);
    c(1):=c(1)-equat1/df(equat1,c(1));
    for j:=2:k do
  <<
    equatpr:=part(equat,j);
    if(abs(equatpr)>10**(-11)) then
    << clear c(2*j-1);
       test:=0;
       equatpr:=part(equat,j);
       respr:=solve(equatpr,c(2*j-1));
       while respr neq {} do
       << c(2*j-1):=part(respr,1,2);
          if(c(2*j-1)=sub(i=-i,c(2*j-1)) and c(2*j-1)<1 )
            then respr:={}
            else respr:=rest(respr);
       >>;
    >>;
  >>;
 >>;
>>;

% Write out the obtained result.

write cw:=c(1);
c(1):=1$
for j:=1:n do  write "c(",2*j-1,"):=", c(2*j-1);
for j:=2:n do  write "c(",2*j-3,")/c(",2*j-1,"):= ", c(2*j-3)/c(2*j-1);
c(1):=cw;
for j:=1:3*n do  write "equ(",j,"):=",test1:=part(equat,j);
quit;
```