Systems of Nonlinear and Algebraic Nonlinear Ordinary Differential Equations

GIUSEPPA CARRA' FERRO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CATANIA
Viale Andrea Doria 6, 95125 CATANIA
ITALY
carra@dipmat.unict.it

January 19, 1999

Abstract

A procedure is shown in order to study a special class of ordinary nonlinear and nonalgebraic differential equations. In fact ordinary differential equations, that are polynomials in the derivatives and in some elementary functions of the derivatives are considered. A procedure it is shown in order to transform such equations in algebraic differential equations by using some differential elimination procedures.

1 Introduction

Let t be an independent variable and let $y = y(t)$ be a dependent variable. Let $y^{(n)}$ be the derivative of order n of y with respect to t. Let

$$P = P(y, y^{(1)}, \dots, y^{(n)}, z_1, \dots, z_h, v_1, \dots, v_k)$$

be a polynomial in the variables $y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k$ with coefficients in a differential field F of functions, as example $F = \mathbf{R}(t)$ field of rational functions with coefficients in $\mathbf{R}$.

Let $z_i = exp(Q_i)$ and let $v_j = log(R_j)$, where $Q_i = Q_i(y, y^{(1)}, \ldots, y^{(q_i)})$ and $R_j = R_j(y, y^{(1)}, \ldots, y^{(r_j)})$ are polynomial functions with coefficients in the same differential field F for all $i = 1, \ldots, h$ and all $j = 1, \ldots, k$.
The ordinary differential equation

$$\{P = P(y, y^{(1)}, \ldots, y^{(n)}, exp(Q_1), \ldots, exp(Q_h), log(R_1), \ldots, log(R_k)) = 0\}$$

is in general nonlinear and nonalgebraic.

The equation $\{P = 0\}$ is equivalent to the system

$$\begin{gathered}\{P = P(y, y^{(1)}, \ldots, y^{(n)}, exp(Q_1), \ldots, exp(Q_h), log(R_1), \ldots, log(R_k)) = 0, \\ P^{(1)} = 0,\ \ldots,\ P^{(m)} = 0,\ z_i = exp(Q_i),\ v_j = log(R_j), \\ z_i^{(1)} = Q_i^{(1)} exp(Q_i) = Q_i^{(1)} z_i,\ R_j v_j^{(1)} = R_j^{(1)},\ i = 1, \ldots, h,\ j = 1, \ldots, k, \\ m \in \mathbf{N}_0\},\end{gathered}$$

where $P^{(m)}$ is the derivative of order m of the function $P = P(t)$ with respect to t for all $m \geq 1$.

Let $P^{(1)} = P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)^{(1)} =$
$P^{(1)}(y, y^{(1)}, \ldots, y^{(n+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_k^{(1)})$.

By algebraically eliminating $v_k^{(1)}$ between $P^{(1)}$ and $R_k v_k^{(1)} - R_k^{(1)}$ and by using equalities $z_i^{(1)} = Q_i^{(1)} z_i$ a polynomial

$$P_1' = P_1'(y, y^{(1)}, \ldots, y^{(N+1)}, z_1, \ldots, z_h, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_{k-1}^{(1)})$$

is obtained, where $N = \max\{n, q_1, \ldots, q_h, r_1, \ldots, r_k\}$.

By algebraically eliminating v_k between P and P_1' a polynomial $P_1 = P_1(y, y^{(1)}, \ldots, y^{(N+1)}, z_1, \ldots, z_h, v_1, \ldots, v_{k-1})$ is obtained.

By using this procedure we can eliminate $z_1, \ldots, z_h, v_1, \ldots, v_{k-1}$ and we obtain a polynomial P_{h+k} only in y and its derivatives, such that every solution of $\{P = 0\}$ is a solution of $\{P_{h+k} = 0\}$.

The same procedure can be applied to dynamical systems.

There are many differential elimination procedures. Some of them are based on the characteristic set theory of Wu-Ritt [*Ritt (1950, Kolchin (1973)*]

[*Wu (1987)*, *Wu (1989)*, *Wang (1993)*]; other procedures use in addition the Gröbner bases theory for polynomial ideals [*Boulier (1994)*, *Boulier (1996)*] [*Boulier and Lazard and Ollivier and Petitot (1995)*, *Diop (1991)*] [*Diop (1991)*, *Diop (1991)*, *Diop (1992)*, *Diop and Fliess (1991)*] [*Fliess (1989)*, *Fliess (1990)*], some others are based on the differential resultants theory [*Carra' Ferro (1997)*, *Carra' Ferro (1997)*] and the differential Gröbner bases theory for differential polynomial ideals [*Carra' Ferro (1987)*, *Ollivier (1990)*, *Weispfenning (1993)*].

2 Differential Algebra Preliminaries

We assume that everything, which is undefined, is as in [*Ritt (1950*] and in [*Kolchin (1973)*]. Let δ be a derivation operator and let R be a differential ring, i.e. a commutative ring with unit and a derivation δ acting on it, such that the field of rational numbers $\mathbf{Q}$ is contained in R.

A differential field F is a field, which is also a differential ring. Let C_R=$\{a \in R\text{: } \delta(a) = 0\}$ be the ring of constants of R and let $\mathbf{N}_0 = \{0, 1, \ldots, n, \ldots\}$.

REMARK 1 $\mathbf{Q} \subseteq C_R$ *because* $\mathbf{Q} \subseteq R$.

EXAMPLE 1 R=$C^\infty(\mathbf{R})$, R=$\mathbf{Q}(t)$, R=$\mathbf{R}(t)$ *and* R=*field of the meromorphic functions on a domain of* $\mathbf{C}$ *with the usual derivation* δ=$\frac{\partial}{\partial t}$ *are differential rings.* R=$C^m(\mathbf{R})$ *for all* $m \in \mathbf{N}_0$ *is not a differential ring.*

If $a \in \mathbf{N}_0$, then *order* of δ^a is $\mathrm{ord}(\delta^a)$=a.

DEFINITION 1 S=$R\{y_1, \ldots, y_m\}$=$R[\delta^n y_i\text{: } i = 1, \ldots, m \text{ and } n \in \mathbf{N}_0]$ *is the differential ring of the differential polynomials in the differential indeterminates* $y_1, \ldots, y_m$ *with coefficients in the differential ring* R.

If $f \in S$, then *order* of f is $\mathrm{ord}(f)$=max$\{$ $n \in \mathbf{N}_0$: f contains a power product in $y_1, \ldots, y_m, \delta y_1, \ldots, \delta y_m, \ldots, \delta^n y_1, \ldots, \delta^n y_m$ with nonzero coefficient $\}$.

DEFINITION 2 *An ideal I of S is a differential ideal iff $\delta^n(s) \in I$ for all $n \in \mathbf{N}_0$ and all $s \in I$. If $A \subseteq S$, then $[A]$=($\delta^n(s)$: $s \in A$, $n \in \mathbf{N}_0$) denotes the smallest differential ideal containing A.*

Let R be a differential ring of functions in the independent variable t, let $\delta=\frac{\partial}{\partial t}$ and let $y=y(t)$ be a differential indeterminate. $y^{(n)}=\delta^n y$ is the derivative of order n of y with respect to t for all positive integers n, while $y^{(0)}=\delta^0 y=y$.

So an ordinary algebraic differential equation in m dependent variables $y_1, \ldots, y_m$ with coefficients in R is nothing else than the equality $\{f = 0\}$ for some differential polynomial f in S.

3 Nonalgebraic Ordinary Differential Equations

Let $P=P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)$ be a polynomial in the variables $y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k$ with coefficients in a differential ring R of functions. So $P \in R\{y, z_1, \ldots, z_h, v_1, \ldots, v_k\}$.

Let $z_i=exp(Q_i)$ and let $v_j=log(R_j)$, where $Q_i=Q_i(y, y^{(1)}, \ldots, y^{(q_i)})$ and $R_j=R_j(y, y^{(1)}, \ldots, y^{(r_j)})$ are polynomial functions with coefficients in the differential ring R for all $i = 1, \ldots, h$ and all $j = 1, \ldots, k$, e.g. $Q_i, R_j \in R\{y\}$ for all i and j. Moreover $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

The ordinary differential equation

$$\{P = P(y, y^{(1)}, \ldots, y^{(n)}, exp(Q_1), \ldots, exp(Q_h), log(R_1), \ldots, log(R_k)) = 0\}$$

is in general nonlinear and nonalgebraic.

Since $exp(a)$ (respectively $log(a)$) with a in a differential ring R are uniquely defined by the differential equations $\{ y^{(1)} - a^{(1)}y=0 \}$ (respectively $\{ ay^{(1)} - a^{(1)}=0 \}$) up to a constant, then the equation {P=0} is equivalent to the system

(α)=\{ $P=P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)=0$, $P^{(1)}=0, \ldots$, $P^{(m)}=0$, $z_i=exp(Q_i)$, $v_j=log(R_j)$, $z_i^{(1)}= Q_i^{(1)} exp(Q_i)=Q_i^{(1)} z_i$, $R_j v_j^{(1)}=R_j^{(1)}$, $i = 1, \ldots, h$, $j = 1, \ldots, k$, $m \in \mathbf{N}_0$ \}, where $P^{(m)}$ is the derivative of order m of P for all $m \geq 1$.

EXAMPLE 2 *Let*

$$P = P(y, y^{(1)}, y^{(2)}, exp(Q_1), exp(Q_2), log(R_1) =$$
$$yy^{(2)} - exp(y + (y^{(1)})^2) log(y + y^{(3)}) y^{(1)} - (exp(y - y^{(2)}))^2 y^2 (y^{(1)})^3,$$

where $Q_1 = y + (y^{(1)})^2$, $Q_2 = y - y^{(2)}$ *and* $R_1 = y + y^{(3)}$.
If $z_1 = exp(Q_1)$, $z_2 = exp(Q_2)$ *and* $v_1 = log(R_1)$, *then*

$$P = yy^{(2)} - z_1 v_1 y^{(1)} - z_2^2 y^2 (y^{(1)})^3.$$

The equation $\{ P=0 \}$ *is equivalent to the system* $(\alpha)=\{ P=yy^{(2)} - z_1 v_1 y^{(1)} - z_2^2 y^2 (y^{(1)})^3=0,\ P^{(1)}=0, \ldots, P^{(m)}=0,\ z_1=exp(y + (y^{(1)})^2),\ z_2=exp(y - y^{(2)}))^2),\ v_1=log(y + y^{(3)}),\ z_1^{(1)}=(y^{(1)} + 2y^{(1)}y^{(2)})z_1,\ z_2^{(1)}= (y^{(1)} - y^{(3)})z_2,\ (y + y^{(3)})v_1^{(1)}=y^{(1)} + y^{(4)},\ m \in \mathbf{N}_0\}$.

Let (β) be the following system of algebraic differential equations

$$(\beta)=\{ P=P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)=0,\ P^{(1)}=0, \ldots, P^{(m)}=0,\ z_i^{(1)}=Q_i^{(1)} z_i,\ R_j v_j^{(1)}=R_j^{(1)},\ i = 1, \ldots, h,\ j = 1, \ldots, k, m \in \mathbf{N}_0 \},$$

with $P \in F\{y, z_1, \ldots, z_h, v_1, \ldots, v_k\}$, $Q_i, R_j \in F\{y\}$ for all i and j.
It is possible to find a system (γ) of algebraic ordinary differential equations in the same differential variables $y, z_1, \ldots, z_h, v_1, \ldots, v_k$, that is equivalent to (β), such that it contains a differential polynomial equation in the differential variable y. In other words we can find a different set of differential generators of the differential ideal $I=[P,\ z_i^{(1)} - Q_i^{(1)} z_i,\ R_j v_j^{(1)} - R_j^{(1)},\ i = 1, \ldots, h,\ j = 1, \ldots, k]$, that has the riquired properties.

3.1 An Elimination Procedure

Let R be a differential ring and let (η) be the following system of ordinary algebraic differential equations:

$$(\eta)=\{ P=P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)=0,\ z_i^{(1)} - Q_i^{(1)} z_i=0, \\ R_j v_j^{(1)} - R_j^{(1)}=0,\ i = 1, \ldots, h,\ j = 1, \ldots, k \}.$$

with $P \in R\{y, z_1, \ldots, z_h, v_1, \ldots, v_k\}$, $Q_i, R_j \in R\{y\}$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

First of all in the following subsubsection an elimination procedure of one differential variable will be shown.

3.1.1 An Elimination Procedure of One Differential Variable

Let $N=\max\{n,q_i,r_j\colon\ i = 1, \ldots, h,\ j = 1, \ldots, k\}$.

Let $P^{(1)} = P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)^{(1)} = P^{(1)}(y, y^{(1)}, \ldots, y^{(N)}, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_k^{(1)})$ and let J_1 be the ideal $(P,\ P^{(1)},\ z_i^{(1)} - Q_i^{(1)} z_i,\ R_j v_j^{(1)} - R_j^{(1)},\ i = 1, \ldots, h,\ j = 1, \ldots, k)$ in the polynomial ring $R[y, \ldots, y^{(N)}, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_k^{(1)}]$.

Let σ_1 be the lexicographic term ordering on the set of power products in $\{y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_k^{(1)}\}$, with $y < \ldots < y^{(N)} < y^{(N+1)} < z_1 < z_1^{(1)} < \ldots < z_h < z_h^{(1)} < v_1 < v_1^{(1)} < \ldots < v_k < v_k^{(1)}$.

Let G_1 be the reduced Gröbner basis of the ideal J_1 with respect to σ_1 [*Buchberger (1976), Buchberger (1976)*]. Since $P^{(1)}= \frac{\partial}{\partial v_k^{(1)}}(P^{(1)}) v_k^{(1)} + T_1$ and $R_k v_k^{(1)} - R_k^{(1)}$ have degree one in $v_k^{(1)}$, then P and $S_1=R_k T_1 + \frac{\partial}{\partial v_k^{(1)}}(P^{(1)}) R_k^{(1)}$ are in $J_1 \cap R[y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_{k-1}^{(1)}]$. So it is possible to eliminate $v_k^{(1)}$ between $P^{(1)}$ and $R_k v_k^{(1)} - R_k^{(1)}$ and by using equalities $z_i^{(1)} = Q_i^{(1)} z_i$ a polynomial $P_1'=P_1'(y, y^{(1)}, \ldots, y^{(N+1)}, z_1, \ldots, z_h, v_1, \ldots, v_k, v_1^{(1)}, \ldots, v_{k-1}^{(1)})$ is obtained with $P_1' \neq P$.

If v_k does not appear in P_1', then P_1' is in $J_1 \cap R[y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, v_1, \ldots, v_{k-1}, v_1^{(1)}, \ldots, v_{k-1}^{(1)}]$. If v_k appears in P_1', then $1 \leq deg_{v_k} P_1' \leq deg_{v_k} P$ and the resultant of P and P_1' with respect to the variable v_k is in $J_1 \cap F[y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_{k-1}, v_1^{(1)}, \ldots, v_{k-1}^{(1)}]$. So there exists at least one differential polynomial P_1 in $G_1 \cap F[y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_{k-1}, v_1^{(1)}, \ldots, v_{k-1}^{(1)}]$ by definition of Gröbner basis with respect to a lexicographic term ordering.

Let $G_{11}=G_1 \cap R[y, \ldots, y^{(N+1)}, z_1, \ldots, z_h, z_1^{(1)}, \ldots, z_h^{(1)}, v_1, \ldots, v_{k-1}, v_1^{(1)}, \ldots, v_{k-1}^{(1)}]$. Since G_{11} is the reduced Gröbner basis of $J_{11}=J_1 \cap R[y, \ldots, y^{(N+1)},$

$z_1,\ldots,z_h,z_1^{(1)},\ldots,z_h^{(1)},v_1,\ldots,v_{k-1},v_1^{(1)},\ldots,v_{k-1}^{(1)}]$ with respect to σ_1 and $z_i^{(1)}-Q_i^{(1)}z_i, R_jv_j^{(1)}-R_j^{(1)} \in J_{11}$ for all $i=1,\ldots,h$, $j=1,\ldots,k-1$, then $P_1 \in G_{11}$ and $P_1{=}P_1(y,\ldots,y^{(N+1)},z_1,\ldots,z_h,v_1,\ldots,v_{k-1},v_1^{(1)},\ldots,v_{k-1}^{(1)})$. If $deg_{v_{k-1}^{(1)}} P_1{=}d$, then the pseudoremainder P_1^* of P_1 with respect to $R_{k-1}v_{k-1}^{(1)}-R_{k-1}^{(1)}$ as polynomials in $v_{k-1}^{(1)}$ is in J_{11} and $v_{k-1}^{(1)}$ does not appear in P_1^*. So we can assume that $P_1{=}P_1(y,\ldots,y^{(N+1)},z_1,\ldots,z_h,v_1,\ldots,v_{k-1},v_1^{(1)},\ldots,v_{k-2}^{(1)})$.

REMARK 2 *The elimination of the differential variable v_k in the system (β) can be also obtained in the following ways.*

(i) By using the Ritt process of differential reduction for the extended characteristic set, it is sufficient to find the extended characteristic set of the differential polynomials P and $R_kv_k^{(1)}-R_k^{(1)}$. It riquires the pseudo division of $P^{(1)}$ by $R_kv_k^{(1)}-R_k^{(1)}$ as polynomials in $R[y,\ldots,y^{(N+1)},z_1,\ldots,z_h,z_1^{(1)},\ldots,z_h^{(1)},v_1,\ldots,v_k,v_1^{(1)},\ldots,v_{k-1}^{(1)}][v_k^{(1)}]$ and the extended characteristic set of the partial remainder of the above pseudodivision R_1 and P [Ritt (1950].

(ii) By using the differential resultant theory, e.g. by finding the differential resultant of the differential polynomials P and $R_kv_k^{(1)}-R_k^{(1)}$, as differential polynomials in $R\{y,z_1,\ldots,z_h,v_1,\ldots,v_{k-1}\}\{v_k\}$. By differential resultant theory such differential resultant is equal to the resultant of the polynomials P, $P^{(1)}$ and $R_kv_k^{(1)}-R_k^{(1)}$ in the polynomial ring $R[y,\ldots,y^{(N+1)},z_1,\ldots,z_h,z_1^{(1)},\ldots,z_h^{(1)},v_1,\ldots,v_{k-1},v_1^{(1)},\ldots,v_{k-1}^{(1)}][v_k,v_k^{(1)}]$ [Carra' Ferro (1997)].
In all such elimination procedures of one differential variable the differential polynomials P, $P^{(1)}$ and $R_kv_k^{(1)}-R_k^{(1)}$ are involved.

(iii) By using the differential Gröbner basis theory with respect to an elimination differential term ordering [Carra' Ferro (1987), Ollivier (1990)] [Weispfenning (1993)].

REMARK 3 *The same elimination procedures hold if the variable z_h has to be eliminated in the system (β). It is sufficient to work with the differential polynomials P, $P^{(1)}$ and $z_h^{(1)}-Q_h^{(1)}z_h$.*

3.1.2 An Elimination Procedure of Finitely Many Differential Variables

Let (η) be the system of ordinary algebraic differential equations:

$$(\eta)=\{\ P=P(y,y^{(1)},\dots,y^{(n)},z_1,\dots,z_h,v_1,\dots,v_k)=0, z_i^{(1)}-Q_i^{(1)}z_i=0,\\ R_jv_j^{(1)}-R_j^{(1)}=0,\ i=1,\dots,h,\ j=1,\dots,k\ \}.$$

with $P\in F\{y,z_1,\dots,z_h,v_1,\dots,v_k\}$, $Q_i,R_j\in F\{y\}$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

Let $N=\max\{n,q_i,r_j\colon\ i=1,\dots,h,\ j=1,\dots,k\}$.

By using an elimination procedure of the variable v_k the system (η) becomes equivalent to the system

$$(\eta_1)=\{\ P=0,\ P^{(1)}=0,\ P_1=0,\ z_i^{(1)}-Q_i^{(1)}z_i=0,\\ R_jv_j^{(1)}-R_j^{(1)}=0,\ i=1,\dots,h,\ j=1,\dots,k\ \}.$$

with $P\in F\{y,z_1,\dots,z_h,v_1,\dots,v_k\}$, $P_1\in F\{y,z_1,\dots,z_h,v_1,\dots,v_{k-1}\}$, Q_i and R_j in $F\{y\}$, $ord(P_1)=N+1$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

By using an elimination procedure of the variable v_{k-1} the system (η_1) becomes equivalent to the system

$$(\eta_2)=\{\ P=0,\ P^{(1)}=0,\ P_1=0,\ P_1^{(1)}=0,\ P_2=0,\ z_i^{(1)}-Q_i^{(1)}z_i=0,\\ R_jv_j^{(1)}-R_j^{(1)}=0,\ i=1,\dots,h,\ j=1,\dots,k\ \}.$$

with $P\in F\{y,z_1,\dots,z_h,v_1,\dots,v_k\}$, $P_1\in F\{y,z_1,\dots,z_h,v_1,\dots,v_{k-1}\}$, $P_2\in F\{y,z_1,\dots,z_h,v_1,\dots,v_{k-2}\}$, $Q_i,R_j\in F\{y\}$, $ord(P_1)=N+1$, $ord(P_2)=N+2$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

Finally by using recursively an elimination procedure of the variables $v_{k-2},\dots,v_1,z_h,\dots,z_{h-i}$ the system (η) is equivalent to the system

$$(\eta_{k+i})=\{\ P=0,\ P^{(1)}=0,\ P_1=0,\ P_1^{(1)}=0,\ \dots,\ P_k=0,\ P_k^{(1)}=0,\dots,\\ P_{k+i-1}=0,\ P_{k+i-1}^{(1)}=0,\ P_{k+i}=0,\ z_i^{(1)}-Q_i^{(1)}z_i=0,\ R_jv_j^{(1)}-R_j^{(1)}=0,\\ i=1,\dots,h,\ j=1,\dots,k\ \}.$$

with $P\in F\{y,z_1,\dots,z_h,v_1,\dots,v_k\}$, $P_1\in F\{y,z_1,\dots,z_h,v_1,\dots,v_{k-1}\},\dots,$ $P_k\in F\{y,z_1,\dots,z_h\},\dots,$ $P_{k+i-1}\in F\{y,z_1,\dots,z_{h-i}\}$, $P_{k+i}\in F\{y,z_1,\dots,$

$z_{h-i-1}\}$, $Q_i, R_j \in F\{y\}$, $ord(P_1)=N+1,\ldots$, $ord(P_k)=N+k,\ldots$, $ord(P_{k+i-1})=N+k+i-1$, $ord(P_{k+i})=N+k+i$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

If $i=h$, then the system (η) is equivalent to the system

$$(\eta_{k+h})=\{\ P=0,\ P^{(1)}=0,\ P_1=0,\ P_1^{(1)}=0,\ \ldots,\ P_k=0,\ P_k^{(1)}=0,\ldots,\\ P_{k+h-1}=0,\ P_{k+h-1}^{(1)}=0,\ P_{k+h}=0,\ z_i^{(1)}-Q_i^{(1)}z_i=0,\ R_j v_j^{(1)}-R_j^{(1)}=0,\\ i=1,\ldots,h,\ j=1,\ldots,k\ \}.$$

with $P \in F\{y, z_1, \ldots, z_h, v_1, \ldots, v_k\}$, $P_1 \in F\{y, z_1, \ldots, z_h, v_1, \ldots, v_{k-1}\},\ldots$, $P_k \in F\{y, z_1, \ldots, z_h\},\ldots$, $P_{k+h-1} \in F\{y, z_1\}$, $P_{k+h} \in F\{y\}$, $Q_i, R_j \in F\{y\}$, $ord(P_1)=N+1,\ldots$, $ord(P_k)=N+k$, $\ldots$, $ord(P_{k+h-1})=N+k+h-1$, $ord(P_{k+h})=N+k+h$, $ord(Q_i)=q_i$ and $ord(R_j)=r_j$ for all i and j.

Now the differential ideal $I=[P,\ z_i^{(1)}-Q_i^{(1)}z_i,\ R_j v_j^{(1)}-R_j^{(1)},\ i=1,\ldots,h,\ j=1,\ldots,k]=[P,\ P_1=0,\ldots,\ P_k,\ldots,P_{k+h-1},\ P_{k+h}=0,\ z_i^{(1)}-Q_i^{(1)}z_i,\ R_j v_j^{(1)}-R_j^{(1)},\ i=1,\ldots,h,\ j=1,\ldots,k]$ so the systems (β) and (η) are equivalent.

Let (γ') be the following system

$$(\gamma')=\{\ P=0,\ P^{(1)}=0,\ P_1=0,\ P_1^{(1)}=0,\ \ldots,\ P_k=0,\ P_k^{(1)}=0,\ldots,\\ P_{k+h-1}=0,\ P_{k+h-1}^{(1)}=0,\ P_{k+h}=0,\ z_i=exp(Q_i),\ v_j=log(R_j),\\ z_i^{(1)}-Q_i^{(1)}z_i=0,\ R_j v_j^{(1)}-R_j^{(1)}=0,\ i=1,\ldots,h,\ j=1,\ldots,k\ \}.$$

The system (γ') is equivalent to the system (α).

Since each elimination of one variable among z_i and v_j riquires one more derivative of P by 3.1.1., then $P_1, \ldots, P_{h+k} \in I$ and the system (γ') is equivalent to the following system (γ):

$$(\gamma)=\{\ P=0,\ P^{(1)}=0,\ldots,\ P^{(h+k)}=0,\ z_i=exp(Q_i),\ v_j=log(R_j),\\ z_i^{(1)}-Q_i^{(1)}z_i=0,\ R_j v_j^{(1)}-R_j^{(1)}=0,\ i=1,\ldots,h,\ j=1,\ldots,k\ \}.$$

Finally the differential equation $\{\ P=P(y, y^{(1)}, \ldots, y^{(n)}, exp(Q_1), \ldots, exp(Q_h), log(R_1), \ldots, log(R_k))=0\ \}$ is equivalent to the system

$$(\alpha')=\{\ P=P(y, y^{(1)}, \ldots, y^{(n)}, z_1, \ldots, z_h, v_1, \ldots, v_k)=0,\ P_{h+k}=\\ P_{h+k}(y, y^{(1)}, \ldots, y^{(N+h+k)})=0,\ z_i=exp(Q_i),\ v_j=log(R_j),\\ i=1,\ldots,h,\ j=1,\ldots,k,\ \},$$

REMARK 4 *The elimination of the differential variables* $z_1, \ldots, z_h, v_1, \ldots, v_k$ *in the system* (β) *can be also obtained in the following ways.*

(i) By using the Ritt's theory of characteristic sets it is sufficient to find the unique differential polynomial $P_{h+k} \in R\{y\}$ *in the extended characteristic set of the set of differential polynomials* $\{P, z_i^{(1)} - Q_i^{(1)} z_i, R_j v_j^{(1)} - R_j^{(1)}: l = 1, \ldots, h+k,\ i = 1, \ldots, h,\ j = 1, \ldots, k\}$ *with respect to a ranking* O *of the differential variables, such that* $y <_O < z_1 <_O \ldots <_O z_h <_O v_1 <_O \ldots <_O v_k$.

(ii) By using the Gröbner bases techniques, it is sufficient to find the Gröbner basis G *of the ideal* $J_{h+k}=(P, P^{(l)}, z_i^{(1)} - Q_i^{(1)} z_i, (z_i^{(1)} - Q_i^{(1)} z_i)^{(l)}, R_j v_j^{(1)} - R_j^{(1)}, (R_j v_j^{(1)} - R_j^{(1)})^{(l)}: l = 1, \ldots, h+k,\ i = 1, \ldots, h,\ j = 1, \ldots, k)$ *in* $R[y, \ldots, y^{(N+h+k)}, z_1, \ldots, z_1^{(h+k)}, \ldots, z_h, \ldots, z_h^{(h+k)}, v_1, \ldots, v_1^{(h+k)}, \ldots, v_k, \ldots, v_k^{(h+k)}]$ *with respect to the lexicographic term ordering* σ *with* $y <_\sigma \ldots <_\sigma y^{(N+h+k)} <_\sigma z_1 <_\sigma \ldots <_\sigma z_1^{(h+k} <_\sigma \ldots <_\sigma z_h <_\sigma \ldots <_\sigma z_h^{(h+k} <_\sigma v_1 <_\sigma \ldots <_\sigma v_1^{(h+k} <_\sigma \ldots <_\sigma v_k <_\sigma \ldots <_\sigma v_k^{(h+k}$. *The riquired differential polynomial* P_{h+k} *is in* $G \cap R[y, \ldots, y^{(N+h+k)}]$.

(iii) By using the differential resultant theory, e.g. by finding the differential resultant of the $h+k+1$ *differential polynomials* P, $z_1^{(1)} - Q_1^{(1)} z_1, \ldots, z_h^{(1)} - Q_h^{(1)} z_h$, $R_1 v_1^{(1)} - R_1^{(1)}, \ldots, R_k v_k^{(1)} - R_k^{(1)}$, *as differential polynomials in* $R\{y\}\{z_1, \ldots, z_h, v_1, \ldots, v_{k-1}, v_k\}$.
Since $ord(P)=n$, $ord(Q_i)=q_i$ *and* $ord(R_j)=r_j$ *for all* i *and* j, *let* $N^*=n + \sum_{i=1,\ldots,h} max(1, q_i) + \sum_{j=1,\ldots,k} max(1, r_j)$. *By differential resultant theory such differential resultant is equal to the resultant of the polynomials* $P, \ldots, P^{(N^*-ord(P))}$, $z_i^{(1)} - Q_i^{(1)} z_i, \ldots, (z_i^{(1)} - Q_i^{(1)} z_i)^{(N^*-max(1,q_i))}, R_j v_j^{(1)} - R_j^{(1)}, \ldots, (R_j v_j^{(1)} - R_j^{(1)})^{(N^*-max(1,r_j))}$, $i = 1, \ldots, h,\ j = 1, \ldots, k)$ *in the polynomial ring* $R[y, \ldots, y^{(N^*)}, z_i, \ldots, z_i^{(N^*-max(1,q_i))}, v_j, \ldots, v_j^{(N^*-max(1,r_j))}]$: $i = 1, \ldots, h$, $j = 1, \ldots, k]$.
Since each elimination process of the above differential variables riquires at most $h+k$ *derivatives of* P, $z_i^{(1)} - Q_i^{(1)} z_i$, $R_j v_j^{(1)} - R_j^{(1)}$, *then this elimination process holds also when* $P \in C^{r+h+k}(\mathbf{R})$ *for some* $r \in \mathbf{N}_0$.

3.2 Examples

The following examples were found with the help of the system Maple V Release 4.

EXAMPLE 3 *Let* $P = P(y, y^{(1)}, exp(Q_1)) = y^{(1)} - exp(y^{(1)} - y)$, *where* $Q_1 = y^{(1)} - y$. *By using the command* dsolve *there is no answer to* $dsolve(P = 0, y)$. *If* $z = exp(Q_1)$, *then the equation* $\{P = 0\}$ *is equivalent to the system* $(\alpha) = \{y^{(1)} - z = 0,\ z^{(1)} - y^{(2)} + y^{(1)} = 0,\ y^{(2)} - z^{(1)} = 0,\ z = exp(y^{(1)} - y)\}$.
By eliminating z, *the system* (α) *is equivalent to the system* $(\beta) = \{y^{(1)} - z = 0,\ z^{(1)} - y^{(2)} + y^{(1)} = 0,\ P_1 = -y^{(2)} + y^{(1)}y^{(2)} - (y^{(1)})^2 = 0,\ z = exp(y^{(1)} - y)\}$.
By using the command dsolve, $dsolve(P_1 = 0, y)$ *gives the answer*

$$y = log(LambertW(-exp(-t + c_1)) - \frac{1}{LambertW(-exp(-t + c_1))} + c_2,$$

where c_1, c_2 *are arbitrary constants and* $LambertW$ *is a solution of the equation* $yexp(y) - t = 0$. *The solutions of* $\{P = 0\}$ *are given by substituting* y *and its derivatives in* $P = 0$ *by the corresponding solutions of* $P_1 = 0$ *in order to find conditions on the constants, because there is only one constant in the general solution of* $P = 0$.

EXAMPLE 4 *Let* $P = P(y, y^{(1)}, log(R_1)) = y^{(1)} - log(y^{(1)} - y)$, *where* $R_1 = y^{(1)} - y$. *By using the command* dsolve *there is no answer to* $dsolve(P = 0, y)$. *If* $v = log(R_1)$, *then the equation* $\{P = 0\}$ *is equivalent to the system* $(\alpha) = \{y^{(1)} - v = 0,\ y^{(1)}v^{(1)} - yv^{(1)} - y^{(2)} + y^{(1)} = 0,\ y^{(2)} - v^{(1)} = 0,\ v = log(y^{(1)} - y)\}$.
By eliminating v, *the system* (α) *is equivalent to the system* $(\beta) = \{y^{(1)} - v = 0,\ y^{(1)}v^{(1)} - yv^{(1)} - y^{(2)} + y^{(1)} = 0,\ P_1 = -y^{(2)} + y^{(1)} + y^{(1)}y^{(2)} - yy^{(2)}) = 0,\ v = log(y^{(1)} - y)\}$.
By using the command dsolve, $dsolve(P_1 = 0, y)$ *gives the answer*

$$t = \int_0^y \frac{1}{-LambertW(-c_1 exp(y^{(1)}) + y^{(1)})} dy^{(1)} - c_2$$

where c_1, c_2 *are arbitrary constants and* $LambertW$ *is a solution of the equation* $yexp(y) - t = 0$. *The solutions of* $\{P = 0\}$ *are given by substituting*

y and its derivatives in $P=0$ by the corresponding solutions of $P_1=0$ in order to find conditions on the constants, because there is only one constant in the general solution of $P=0$.

EXAMPLE 5 *Let $P{=}P(y,y^{(1)},exp(Q_1),log(R_1)){=}y^{(1)}-y-exp(y^{(1)})-log(y^{(1)})$, where $Q_1{=}y^{(1)}$ and $R_1{=}y^{(1)}$. By using the command* dsolve *there is no answer to $dsolve(P=0,y)$. If $z{=}exp(Q_1)$ and $v{=}log(R_1)$, then the equation $\{P=0\}$ is equivalent to the system*
$(\alpha){=}\{y^{(1)}-y-z-v{=}0,\ z_1-y^{(2)}{=}0,\ y^{(1)}v^{(1)}-y^{(2)}{=}0,\ y^{(2)}-y^{(1)}-z^{(1)}-v^{(1)}{=}0,$
$z{=}exp(y^{(1)}),\ v{=}log(y^{(1)})\}$.
By eliminating v and z, the system (α) is equivalent to the system
$(\beta){=}\{y^{(1)}-v{=}0,\ z_1-y^{(2)}{=}0,\ y^{(1)}v^{(1)}-y^{(2)}{=}0,\ P_1{=}y^{(1)}y^{(2)}z-y^{(1)}y^{(2)}+(y^{(1)})^2+y^{(2)}{=}0,\ P_1^{(1)}{=}0,\ P_2{=}(y^{(2)})^3+(y^{(2)})^3y^{(1)}-(y^{(2)})^2(y^{(1)})^2-(y^{(2)})^3(y^{(1)})^2+(y^{(2)})^2(y^{(1)})^3+y^{(3)}(y^{(1)})^3\}$.
By using the command dsolve*, $dsolve(P_2=0,y)$ gives the answer*

$$y = exp((texp(c)+Ei(-exp(c))c_1esp(c)-1+c_2exp(x))exp(-c)+c_3,$$

where c_1,c_2,c_3 are arbitrary constants and $Ei(t)=\int\frac{exp(-tz)}{z}dt$. The solutions of $\{P=0\}$ are given by substituting y and its derivatives in $P=0$ by the corresponding solutions of $P_1=0$ in order to find conditions on the constants, because there is only one constant in the general solution of $P=0$.

EXAMPLE 6 *Let $P{=}P(y,y^{(1)},exp(Q_1),log(R_1)){=}y^{(1)}-exp(y^{(1)})-log(y^{(1)})$, where $Q_1{=}y^{(1)}$ and $R_1{=}y^{(1)}$. By using the command* dsolve *there is no answer to $dsolve(P=0,y)$. If $z{=}exp(Q_1)$ and $v{=}log(R_1)$, then the equation $\{P=0\}$ is equivalent to the system*
$(\alpha){=}\{y^{(1)}-z-v{=}0,z_1-y^{(2)}{=}0,\ y^{(1)}v^{(1)}-y^{(2)}{=}0,\ y^{(2)}-z^{(1)}-v^{(1)}{=}0,$
$z{=}exp(y^{(1)}),v{=}\ log(y^{(1)})\}$.
By eliminating v and z, the system (α) is equivalent to the system
$(\beta){=}\{y^{(1)}-v{=}0,\ z_1-y^{(2)}{=}0,\ y^{(1)}v^{(1)}-y^{(2)}{=}0,\ P_1{=}y^{(1)}y^{(2)}z-y^{(1)}y^{(2)}+y^{(2)}{=}0,$
$P_1^{(1)}{=}0,\ P_2=-(y^{(2)})^3-(y^{(2)})^3y^{(1)}+(y^{(2)})^3(y^{(1)})^2\}$.
By using the command dsolve*, $dsolve(P_2=0,y)$ gives the answer*

$$y=\frac{1+\sqrt{5}}{2}t+c_1, y=\frac{1-\sqrt{5}}{2}t+c_1, y=c_1t+c_2,$$

where c_1, c_2 *are arbitrary constants. By using the comand* fsolve*,* $fsolve(x - exp(x) - log(x), x)$ *gives the answer* $x = 0.3441612867$*.*
So $y = 0.3441612867t + c_2$ *is the general solution of* $\{P = 0\}$*.*

4 Systems of Nonalgebraic Ordinary Differential Equations

Let R be a differential ring of functions and let $P_l{=}P_l(y_1, \ldots, y_1^{(n_{1l})}, \ldots, y_m, \ldots, y_m^{(n_{ml})}, z_1, \ldots, z_h, v_1, \ldots, v_k)$ in $R\{y_1, \ldots, y_m, z_1, \ldots, z_h, v_1, \ldots, v_k\}$ with $l = 1, \ldots, p$

Let $z_i{=}exp(Q_i)$ and let $v_j{=}log(R_j)$, where $Q_i{=}Q_i(y_1, \ldots, y_1^{(q_{i1})}, \ldots, y_m, \ldots, y_m^{(q_{im})})$ and $R_j{=}R_j(y_1, \ldots, y_1^{(r_{j1})}, \ldots, y_m, \ldots, y_m^{(r_{jm})})$ are polynomial functions with coefficients in the differential ring R for all $i = 1, \ldots, h$ and all $j = 1, \ldots, k$, e.g. $Q_i, R_j \in R\{y_1, \ldots, y_m\}$ for all i and j. The system of ordinary differential equations

$$(\alpha){=}\{\ P_l{=}P_l(y_1, \ldots, y_1^{(n_{1l})}, \ldots, y_m, \ldots, y_m^{(n_{ml})}, exp(Q_1), \ldots, exp(Q_h), log(R_1), \ldots, log(R_k)){=}0,\ l = 1, \ldots, p\ \}$$

is in general a system of nonlinear and nonalgebraic differential equations.

Since $exp(a)$ (respectively $log(a)$) with a in a differential ring R are uniquely defined by the differential equations $\{\ y^{(1)} - a^{(1)}y{=}0\ \}$ (respectively $\{\ ay^{(1)} - a^{(1)}{=}0\ \}$) up to a constant, then the system (α) is equivalent to the system

$$(\beta){=}\{\ P_l{=}P_l(y_1, \ldots, y_1^{(n_{1l})}, \ldots, y_m, \ldots, y_m^{(n_{ml})}, z_1, \ldots, z_h, v_1, \ldots, v_k){=}0,\ P_l^{(1)}{=}0, \ldots,\ P_l^{(m_l)}{=}0,\ z_i{=}exp(Q_i),\ v_j{=}log(R_j),\ z_i^{(1)}{=} Q_i^{(1)}z_i,\ R_jv_j^{(1)}{=}R_j^{(1)},\ l = 1, \ldots, p\ i = 1, \ldots, h,\ j = 1, \ldots, k,\ m_l \in \mathbf{N}_0\ \},$$

where $P^{(m_l)}$ is the derivative of order m_l of P_l for all $m_l \geq 1$.

Let $N{=}\max\{n_{tl}, q_{it}, r_{jt}\colon\ t = 1, \ldots, m,\ l = 1, \ldots, p,\ i = 1, \ldots, h,\ j = 1, \ldots, k\}$. By using results from section 3, the system (α) is equivalent to the system

(α')=\{ P_l=$P_l(y_1,\ldots,y_1^{(n_{1l})},\ldots,y_m,\ldots,y_m^{(n_{ml})},z_1,\ldots,z_h,v_1,\ldots,$ $v_k)$=0, $z_i^{(1)}$=$Q_i^{(1)}z_i$, $R_jv_j^{(1)}$=$R_j^{(1)}$, P_{1l_1}=$P_{1l_1}(y_1,\ldots,y_1^{(N+1)},\ldots,y_m,$ $\ldots,y_m^{(N+1)},z_1,\ldots,z_h,v_1,\ldots,v_{k-1})$=0,$\ldots$, P_{kl_k}=$P_{kl_k}(y_1,\ldots,$ $y_1^{(N+k)},\ldots,y_m,\ldots,y_m^{(N+k)},z_1,\ldots,z_h)$=0,$\ldots$, $P_{(k+h)l_{k+h}}$= $P_{(k+h)l_{k+h}}(y_1,\ldots,y_1^{(N+h+k)},\ldots,y_m,\ldots,y_m^{(N+k+h)})$=0, z_i=$exp(Q_i)$, v_j=$log(R_j)$, $l=1,\ldots,p$, $i=1,\ldots,h$, $j=1,\ldots,k$, $l_r=1,\ldots,$ $r(l)$, $r=1,\ldots,k+h$\},

EXAMPLE 7 *Let $C(p,q)$=$qexp(p)$ be the conservative function and let $D(p,q)$=$p+q$ be the dissipative function. Let's consider the corresponding nonconservative system*

$$(\alpha)=\{p^{(1)}-\frac{\partial}{\partial q}C-\frac{\partial}{\partial p}D=p^{(1)}-exp(p)-1=0,\ q^{(1)}+\frac{\partial}{\partial p}C-\frac{\partial}{\partial q}D=q^{(1)}+qexp(p)-1=0\}.$$

Let $z=exp(p)$. The system (α) is equivalent to the system

$$(\beta)=\{p^{(1)}-z-1=0,q^{(1)}+qz-1=0,\ z-exp(p)=0,\ z^{(1)}-p^{(1)}z=0\}.$$

By eliminating z the system is equivalent to tha system

$$(\gamma)=\{p^{(1)}-z-1=0,\ q^{(1)}+qz-1=0,\ p^{(2)}-(p^{(1)})^2+p^{(1)}=0,\ z-exp(p)=0,\ z^{(1)}-p^{(1)}z=0\}.$$

$$(p(t)=log(\frac{exp(t)}{1-exp(t+c)})+c,$$
$$q(t)=(1-exp(t+c))\int\frac{1}{(1-exp(t+c))^2}+(1-exp(t+c))c_1),$$

is the solution of the system, where c,c_1 are arbitrary constants.

References

[*Boulier and Lazard and Ollivier and Petitot (1995)*] Boulier, F., Lazard, D., Ollivier F., Petitot M. (1995). Representation for the radical of a finitely generated differential ideal. *Proc. of ISSAC'95*. New York: ACM Press, 158–166

[*Boulier (1994)*] Boulier, F. (1994). Etude et Implantation de Quelques Algorithmes en algebre differentielle. *Ph. D. Thesis*. Universite de Lille

[*Boulier (1996)*] Boulier, F. (1996). Some improvements of a lemma of Rosenfeld. Preprint

[*Buchberger (1976)*] Buchberger, B. (1976). A theoretical basis for the reduction of polynomials to canonical form. *ACM SIGSAM BULL.* **10** (3), 19–29

[*Buchberger (1976)*] Buchberger, B. (1976). Some properties of Gröbner bases for polynomial ideals. *ACM SIGSAM BULL.* **10**, 19–24

[*Carra' Ferro (1987)*] Carra' Ferro, G. (1987). Gröbner Bases and Differential Algebra. *Lec. Not. in Comp. Sci.* **356**, 129–140

[*Carra' Ferro (1997)*] Carra' Ferro, G. (1997). A Resultant Theory for the Systems of Two Ordinary Algebraic Differential Equations. *Applicable Algebra in Engineering, Communication and Computing*, 539-560

[*Carra' Ferro (1997)*] Carra' Ferro, G. (1997). A Resultant Theory for Ordinary Algebraic Differential Equations. *Lect. Notes in Comp. Sci.* **1255**, 55–65

[*Diop and Fliess (1991)*] Diop, S., Fliess, M. (1991). On nonlinear observability. *Proc. of the first Europ. Control Conf. 1991*. Paris: Hermes, 152–157

[*Diop (1991)*] Diop, S. (1991). Elimination in control theory. *Math. Control Signal Systems* **4** , 17–32

[*Diop (1991)*] Diop, S. (1991). Finite morphisms of differential algebraic varieties and elimination theory. *Analysis of Controlled Dynamical Systems Proc. of Conference 1990*. Birkhauser , 193–200

[*Diop (1991)*] Diop, S. (1991). Elimination in Control Theory. *Mathematics of Control, Signal and Systems* **4**, 17–42

[*Diop (1992)]* Diop, S. (1992). Differential algebraic decision methods and some application to system theory. *Theoret. Comp. Sci.* **98**, 137–161

[*Fliess (1989)]* Fliess, M. (1989). Automatique et corps differentiels. *Forum Math.* **1**, 227–238

[*Fliess (1990)]* Fliess, M. (1990). Generalized controller canonical forms for linear and nonlinear dynamics. *IEEE Trans. Autom. Control* **35**, 994–1001

[*Kolchin (1973)]* Kolchin, E. (1973). *Differential Algebra and Algebraic Groups*. New York: Academic Press

[*Ollivier (1990)]* Ollivier, F. (1990). Standard Bases of Differential Ideals. *Lec. Not. in Comp. Sci.* **508**, 304–321

[*Ritt (1950]* Ritt, J.F. (1950). *Differential Algebra*. New York: AMS Coll. Publ. **33**

[*Wang (1993)]* Wang, D.M. (1993). Geometric theorem proving with existing technology. *MEDLAR II Deliverable DII.41P*.

[*Weispfenning (1993)]* Weispfenning, V. (1993). Differential-Term Orders. *Proc. of ISAAC'93 Kiev*. New York: ACM Press, 245-253

[*Wu (1987)]* Wu, W.T. (1987). Mechanical Derivation of Newton's Gravitational Laws from Kepler's Laws. *Mathematics Mechanization Research Reprints*. **1**, 53-61

[*Wu (1989)]* Wu, W.T. (1989). On the Foundation of Algebraic Differential Geometry. *Mathematics Mechanization Research Reprints*. **3**, 1-26 *Systems Sciences and Mathematical Sciences*. **2**, 289-312