On Switched Polynomial Systems and Exact Output Tracking

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Abstract

A design method for stabilizing switched hybrid controllers for continuous-time polynomial systems is presented. The approach relies on the use of the symbolic computation package QEPCAD. The results are relevant for design of relay controllers of polynomial systems and the investigation of the exact output tracking problem for a class of non-affine polynomial systems.

1 Introduction

Switching between different controllers to achieve a required behavior is rather common in practical applications. A common way of implementing switching is for example by the use of PID controllers with selectors [1]. This can be used to guarantee that certain process variables are kept within specified bounds. Simple simulation studies and results from optimal control theory such as bang-bang control show that performance can be enhanced by switching between a number of state-feedback laws. A sufficient condition to be able to design control laws based on switching between different controllers, is that we can choose a controller that makes the time derivative of a positive definite function negative along all trajectories in a neighborhood of the desired set-point [14]. Here we give a computational test of this requirement, formulated as a quantifier elimination problem.

For nonlinear systems, switched controllers can be used for stabilization. Brockett has given a necessary condition for when a system with continuously differentiable right hand side can be stabilized by a continuously differentiable state feedback [3, 15]. However, there are many examples of systems that do not satisfy this condition but can be stabilized by a discontinuous state feedback [11].

In output regulation and tracking problems, the zero dynamics [7, 4] plays an important role. Stable zero dynamics is necessary for proper regulation or tracking, since otherwise there are internal variables that exhibit unbounded behavior. There are nonlinear systems whose zero dynamics cannot be stabilized by continuous state feedback but can be stabilized by switched controllers [11, 10]. In this report we approach the problem of analysis and design of discontinuous control laws in a constructive way. Quantifier elimination is used to answer a number of questions. Is it possible to stabilize a system by switching between a number of so called *basic controllers*? Can we obtain stable zero dynamics for the output regulation or tracking problem using a switched controller? Can we obtain an invariant set of states using a switched controller?

The report is organized as follows. In Section 2 we review some basic concepts and definitions related to stability. Since we focus on switched systems we also need to define what we mean by a solution to a system of differential equations with discontinuous right hand sides. Section 3 deals with the problem of stabilizing a polynomial system by switching between a number of continuous controllers. These results are further applied in Section 4 where we show how zero dynamics of a nonlinear system which is not affine in the control can be stabilized by switching between different output zeroing controllers. A number of examples, where we use quantifier elimination software, is presented in Section 5, and Section 6 gives a summary of the report. Throughout the report we utilize quantifier elimination to deal constructively with the computational questions that arise.

2 Preliminaries

Consider a nonlinear system of the following form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are continuous functions with $f(0, 0) = 0$ and $h(0) = 0$. Since we are interested in whether a system can be stabilized by switching between different controllers we introduce a set of so called *basic controllers* of the form

$$u_i(x) = K_i(x), \qquad i = 1, \dots, N \tag{2}$$

where $K_i : \mathbb{R}^n \supset \mathcal{X}_i \to \mathbb{R}$ is continuous, $K_i(0) = 0$, $i = 1, \dots, N$, and $\mathcal{X}_i$ is closed. To determine when to use a basic controller we also introduce a *switching function* $I(x)$ which maps the set of state measurements into the set of integers $\{1, \dots, N\}$. Hence, the switching function induces a partition of the state space with different enumerated components. We use the following notation

$$u(x) = K_{I(x)}(x) \tag{3}$$

for the control law obtained by choosing basic controller with respect to the switching function I. In the sequel we assume that (1) is controlled by the basic controllers (2) with respect to the switching function I. Such systems may arise in cases when the controller consists of a number of relay nonlinearities which can be used to change the structure of the system (1).

Notice that the right hand side of the closed loop system

$$\dot{x} = f(x, K_{I(x)}) \tag{4}$$

is bounded as long as the state belongs to some compact subset $\mathcal{X}$ of $\cup_i \mathcal{X}_i$. Hence, $|\dot{x}|$ is bounded on $\mathcal{X}$.

Sometimes it can be desired to implement a switching controller using some hysteresis elements, for example to avoid sliding modes. Observe that this kind of controllers cannot be described by a switching function since the hysteresis element introduce a discrete state. We will use the term *switching rule* to distinguish these cases.

Here we state a number of definitions and results that we need for investigating stability and stabilizability of switched systems, see for example [8, 9].

Definition 1 *A function* V *is said to be* positive definite *if* $V(0) = 0$ *and* $V(x) > 0$ *for* $x \neq 0$. *A function* W *is called* negative definite *if* $-W$ *is positive definite.*

Definition 2 *A positive definite function* V *is* radially unbounded *if*

$$\lim_{|x| \to \infty} V(x) = \infty.$$

Definition 3 *Consider the system* $\dot{x} = f(x)$, $f(0) = 0$ *and let* $x(t; x_0)$ *denote the solution that starts from the point* x_0. *The origin is*

(i) stable *if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|x_0| < \delta \quad \Longrightarrow \quad |x(t; x_0)| < \epsilon, \, \forall t \geq 0,$$

(ii) (globally) attractive *if there exists* $\Delta > 0$ *(*$\Delta = \infty$*) such that*

$$|x_0| < \Delta \quad \Longrightarrow \quad \lim_{t \to \infty} |x(t; x_0)| = 0,$$

(iii) (globally) asymptotically stable *if it is stable and (globally) attractive.*

If the origin is asymptotically stable it has a *region of attraction*, i.e., a set of initial states Ω such that $x(t) \to 0$ for all $x(0) \in \Omega$.

Definition 4 *For a time invariant dynamic system, an* invariant set $M \subset \mathbb{R}^n$ *is a set such that if* $x(t_0) \in M$ *then* $x(t) \in M$ *for all* $t \geq t_0$.

Note that sometimes the term *positively invariant* is used to describe the above property. However, we do not adopt that terminology here.

Definition 5 *The system* (1) *is* stabilizable *by switching the basic controllers* (2) *if there exists a switching function* I *such that the origin of the closed loop system* (4) *is asymptotically stable.*

The main problem we consider in this report is:

> What are the conditions for existence and how is it possible to design a (globally) stabilizing switching controller (3).

Much of the stability analysis of smooth systems can be generalized to nonsmooth systems.

2.1 Nonsmooth Systems

If we use a piecewise continuous state feedback controller $u = K_{I(x)}(x)$ the right hand side of the closed system (4), $\tilde{f}(x) \triangleq f(x, K_{I(x)}(x))$ becomes discontinuous. There are several ways to define what we mean by *a solution* to such a system. For example it is possible to define solutions according to Filippov [5].

Definition 6 *A function* $x(\cdot)$ *is called a solution of* $\dot{x} = \tilde{f}(x)$ *on* $[t_0, t_1]$ *if* $x(\cdot)$ *is absolutely continuous on* $[t_0, t_1]$ *and for almost all* $t \in [t_0, t_1]$

$$\dot{x} \in F(x), \tag{5}$$

where

$$F(x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{\mathrm{co}}\, \tilde{f}(B_\delta(x) \setminus N). \tag{6}$$

Here $\overline{\mathrm{co}}\, \tilde{f}(x)$ *denotes the convex closure of all limiting values of* $\tilde{f}(x)$*,* $B_\delta(x)$ *is an open hyperball centered at* x *with a radius* δ*, and* $\bigcap_{\mu N=0}$ *is the intersection over all sets* N *of Lebesgue measure zero.*

The definition can be interpreted as follows: the tangent vector to a solution, where it exists, must lie in the convex closure of the limiting values of the vector field in progressively smaller neighborhoods around the solution point.

Note that $F(x)$ in (5) is a set-valued function but corresponds to a singleton in the domains of continuity of $\tilde{f}(x)$.

Solutions to differential inclusions of the type (5) can be shown to exist under rather general conditions [5]. It can be shown that there always exists a solution of the system (1) if we switch among the controllers (2) in a sufficiently regular way. Here we have no intention of going into detail of existence issues. In the sequel we will assume that the chosen switching function or switching rule guarantee that a solution always exist.

Investigations of Lyapunov stability of nonsmooth systems can be found in [5, 13].

3 Stabilizing Polynomial Systems by Switching Controllers

First, we present a general approach to the design of stabilizing switched controllers for systems (1) which are based on the basic controllers (2). It turns out that if we always can choose a basic controller such that a positive definite function decreases along trajectories then the system can be stabilized by switching basic controllers.

To be able to formulate the problem of stabilizing a system by switching basic controllers in a way suitable for quantifier elimination, we will use the following theorem. In the sequel we let $V_x(x) = (\frac{\partial V(x)}{\partial x_1}, \ldots, \frac{\partial V(x)}{\partial x_n})$ denote the gradient of V w.r.t. x.

Theorem 1 *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable positive definite function, and* $\Omega_\alpha = \left\{ x \in \mathbb{R}^n \mid V(x) \leq \alpha \right\}$ *a compact sublevel set of* V*. Suppose* *that for any* $x \in \Omega_\alpha \setminus \{0\}$ *there exists* $i \in \{1, \dots, N\}$ *such that*

$$V_x(x)\, f(x, K_i(x)) < 0. \tag{7}$$

Then the system (1) *is stabilizable by switching the basic controllers* (2) *and* Ω_α *is a region of attraction.*

Proof Introduce the sets

$$S_i = \left\{ x \in \mathbb{R}^n \mid V_x(x) f(x, K_i(x)) < 0 \right\}, \qquad i = 1, \dots, N.$$

Since $V_x(x)f(x, K_i(x))$ is continuous, these sets are open. Let $\epsilon_0 > 0$ and consider $\Omega_\alpha \setminus B_{\epsilon_0}(0)$. According to the assumptions, any $x^* \in \Omega_\alpha \setminus B_{\epsilon_0}(0)$ belongs to some S_i. Since S_i is open, there is an $i \in \{1, \dots, N\}$ and $\epsilon > 0$ such that $B_\epsilon(x^*) \subset S_i$. The collection $\{B_{\epsilon/2}(x^*)\}_{x^* \in \Omega_\alpha \setminus B_{\epsilon_0}(0)}$ forms an open cover of $\Omega_\alpha \setminus B_{\epsilon_0}(0)$. Since $\Omega_\alpha \setminus B_{\epsilon_0}(0)$ is compact, there exists a finite subcover $\{ B_{\epsilon/2}(x_j) \}_{j=1}^M$, $x_j \in \Omega_\alpha \setminus B_{\epsilon_0}(0)$. The collection $\{\overline{B}_{\epsilon/2}(x_j)\}_{j=1}^M$ is also a cover. Now, in each $\overline{B}_{\epsilon/2}(x_j)$ we have $V_x(x)f(x, K_i(x)) < 0$ for at least one of the functions, which also attains its maximum in this set. Hence

$$-\epsilon_1 \triangleq \max_j \min_i \max_{x \in \overline{B}_{\frac{\epsilon}{2}}(x_j)} V_x(x) f(x, K_i(x)),$$

where $\epsilon_1 > 0$. We conclude that for any $\epsilon_0 > 0$ there exists $\epsilon_1 > 0$ such that for any $x \in \Omega_\alpha \setminus B_{\epsilon_0}(0)$ there is an $i \in \{1, \dots, N\}$ such that

$$V_x(x)f(x, K_i(x)) \le -\epsilon_1 < 0. \tag{8}$$

Let $x(t; x_0)$ denote a solution starting at $x_0 \in \Omega_\alpha \setminus B_{\epsilon_0}(0)$ at time 0 of the system (1) controlled by switching basic controllers such that (7) is satisfied. Furthermore, let $\dot{V}(x(t; x_0))$ denote the time derivative of $V(x)$ along a solution. According to (8) we have $\dot{V}(x(t; x_0)) \le -\epsilon_1 < 0$ for all points on the trajectory in $\Omega_\alpha \setminus B_{\epsilon_0}(0)$. Integrating this inequality from 0 to t gives

$$V(x(t; x_0)) \le V(x_0) - \epsilon_1\, t \le \alpha - \epsilon_1\, t. \tag{9}$$

This shows that $x(t; x_0)$ can only stay in $\Omega_\alpha \setminus B_{\epsilon_0}(0)$ for a finite time, since V is bounded from below. Since Ω_α is invariant (V is decreasing along solutions) the trajectory has to enter $B_{\epsilon_0}(0)$. Hence, $\lim_{t \to \infty} x(t; x_0) = 0$ since $\epsilon_0 > 0$ was arbitrary, i.e., the origin is attractive.

To prove stability we observe that Ω_{α^*}, $\alpha^* \le \alpha$ is invariant. Moreover, there exists $d_1(\alpha^*) > 0$ and $d_2(\alpha^*) > 0$ such that $B_{d_1(\alpha^*)} \subsetneq \Omega_{\alpha^*} \subsetneq B_{d_2(\alpha^*)}$, since V is positive definite, see for example [8].

We conclude that the origin is an asymptotically stable solution and Ω_α is a region of attraction of the system (1). □

The assumptions in the theorem implies that the domains of definition of the control laws cover Ω_α, i.e., $\Omega_\alpha \subset \cup_i \mathcal{X}_i$.

Notice that each of the basic controllers may yield an unstable closed loop system but there may still exist a switched stabilizing strategy. The following corollary follows easily from the proof of Theorem 1 since all sublevel sets of a function, which is radially unbounded, are compact.

Corollary 1 *If in addition to the assumptions in the previous theorem, V is radially unbounded, then system* (1) *is globally stabilizable by switching the basic controllers.*

If the assumptions in Theorem 1 or Corollary 1 are satisfied then the following controller can be used to stabilize the system

$$\begin{aligned} u(x) &= K_{I(x)}(x) \\ I(x) &= \underset{i\in\{1,\ldots,N\}}{\operatorname{argmin}}\ V_x(x)\, f(x, K_i(x)). \end{aligned} \tag{10}$$

With this choice we get $\dot{V}(x) = V_x(x)\, f(x, K_{I(x)}(x)) < 0$ for all $x \in \Omega_\alpha \setminus \{0\}$. Observe that the switching function I might be multiple valued since the minimum in (10) might be obtained by several controllers simultaneously. This is the case along switching surfaces. For states corresponding to multiple values the behavior of the controller has to be further specified. Either techniques from sliding mode control [16] can be used or a slight variation of controller (10) where hysteresis is introduced. Due to the fact that the sets S_i (introduced in the proof of Theorem 1) are open, the switching surfaces of controller (10) can be substituted by hysteresis zones. This will prevent sliding modes but chattering solutions may appear instead depending on the direction of the vector fields along the switching surface, see Example 3.

The conditions in Theorem 1 and Corollary 1 can be checked by quantifier elimination if the functions that appear in the theorems can be described by semialgebraic sets. This means that we are able to handle feedback control laws which are implicitly defined as solutions to multivariate polynomial equations. In connection to stabilization of the zero dynamics of a nonlinear system, which is not affine in the control, this turns out to be very useful, see Section 4.

The decision problem that has to be solved to verify the conditions of Theorem 1 is

$$\exists r\ \forall x \Big[\big[x \neq 0 \ \wedge\ V(x) \leq r^2 \big] \ \rightarrow\ \big[\dot{V}_1(x) < 0 \ \vee\ \ldots\ \vee\ \dot{V}_N(x) < 0 \big] \Big] \tag{11}$$

and for Corollary 1 we get

$$\forall x \Big[x \neq 0 \ \rightarrow\ \big[\dot{V}_1(x) < 0 \ \vee\ \ldots\ \vee\ \dot{V}_N(x) < 0 \big] \Big]. \tag{12}$$

If K_i is an algebraic function of x, i.e., if the relation between u_i and x is a polynomial equation $k_i(u_i, x) = 0$ and u_i is one of the roots, the above formulas have to be modified. We have to quantify u and $\dot{V}_i(x) < 0$ is changed to $V_x(x)\, f(x, u_i) < 0\ \wedge\ k_i(u_i, x) = 0$, see Example 4.

Furthermore, given a family of parameterized positive definite functions $\mathcal{V}(x; \theta)$ we can use quantifier elimination to determine if there exists a function in this family such that it satisfies the conditions in Theorem 1 or Corollary 1. For instance, in order to check global stabilizability using the family of quadratic positive definite functions, $V(x) = x^T P x$, we consider the following decision problem

$$\begin{aligned} \exists P\ \forall x \Big[&\big[P^T = P \ \wedge\ \mathrm{Re}(\mathrm{eig}(P)) > 0 \big] \ \wedge \\ &\big[x \neq 0 \ \rightarrow\ \big[\dot{V}_1(x) < 0 \ \vee\ \ldots\ \vee\ \dot{V}_N(x) < 0 \big] \big] \Big], \end{aligned} \tag{13}$$

where P is a matrix variable and $\mathrm{Re}(\mathrm{eig}(P)) > 0$ denotes a set of inequalities which guarantee that P is positive definite, e.g., the positiveness of the principal diagonal minors of P.

This procedure is much more general than when we just use one fixed function $V(x)$. The main drawback of this approach is the large number of variables in the formula. The number of variables is equal to the sum of the number of states, n, and the number of parameters we need to describe $V(x)$. In the quadratic case we get $n + n(n+1)/2 - 1$ variables.

To reduce the required computations, we can search for a quadratic Lyapunov function for the linearized switched dynamics and then use this in the decision problem (11). The following results from [11, 14] are then useful.

Lemma 1 *Consider a linear system*

$$\dot{x} = A\,x + B\,u \tag{14}$$

and a set of linear basic controllers

$$u_i(x) = K_i\,x, \;\; i = 1, 2, \ldots, N \tag{15}$$

If there exist numbers $\tau_i \geq 0, \sum_i \tau_i > 0$ *such that the matrix*

$$\sum_{i=1}^{N} \tau_i (A + B\,K_i)$$

is Hurwitz, then the system (14) is stabilizable by switching the basic controllers (15).

Proof See [14, 10]. □

Lemma 2 *Consider the system* (1) *with basic controllers* (2) *and assume that the system has been rewritten in the following form*

$$\dot{x} = \frac{\partial f(x, K_i(x))}{\partial x}|_{x=0}\; x + g_i(x) = F_i x + g_i(x), \tag{16}$$

where $g_i(x)$ *denotes higher order terms satisfying*

$$\lim_{|x|\to 0} \frac{|g_i(x)|}{|x|} = 0, \qquad i = 1, \ldots, N.$$

If there exist numbers $\tau_i \geq 0, \sum_i \tau_i > 0$ *such that the matrix*

$$\sum_{i=1}^{N} \tau_i F_i \tag{17}$$

is Hurwitz, then the system (1) *is locally stabilizable by switching the basic controllers* (2).

Proof See [11]. □

The results above and in particular the linearization result of Lemma 2 can be used to considerably reduce the required computations of the stabilizability test. Indeed, in order to test stabilizability it is easier if one uses linearizations but the design of a stabilizing controller and the estimation of the domain of attraction can be done directly by using quantifier elimination. We propose the following procedure:

1. Rewrite system (1) in the form (16) for different basic controllers (2). Consider the matrix (17). Find the set of inequalities which follow from the well known Routh-Hurwitz criterion. Computationally more feasible inequalities can be obtained from the Lienard-Chipart criterion, see [6]. The resulting polynomial inequalities in τ_i are denoted $\mathrm{Re}(\mathrm{eig}(\tau_1 F_1 + \ldots + \tau_N F_N) < 0$. Check whether the following decision problem is TRUE

$$\exists \tau \Big[\tau_1 \geq 0 \wedge \ldots \wedge \tau_N \geq 0 \wedge \tau_1 + \ldots + \tau_N > 0 \wedge \mathrm{Re}(\mathrm{eig}(\tau_1 F_1 + \ldots + \tau_N F_N) < 0 \Big]. \qquad (18)$$

   Note that the problem can be normalized by fixing one of the τ_i to 1, which reduces the number of variables.

2. If the above formula is TRUE, the system (1) is stabilizable by switching the basic controllers (2). A numerical instantiation of τ can be extracted from the quantifier elimination procedure. Take any such solution τ. Since the matrix $\sum_i \tau_i F_i$ is Hurwitz, it satisfies the Lyapunov matrix equation. Given any symmetric and positive definite matrix $Q = Q^T, Q > 0$, there exists a unique solution P to

$$(\sum_i \tau_i F_i)^T P + P(\sum_i \tau_i F_i) = -Q, \qquad (19)$$

   which is symmetric and positive definite. Fix the matrix Q and compute P.

3. With the computed P consider the quantifier elimination problem

$$\forall x \Big[[x \neq 0 \wedge x^T P x \leq r^2] \rightarrow [x^T P f(x, K_1(x)) < 0 \vee \ldots \vee x^T P f(x, K_N(x)) < 0] \Big]. \qquad (20)$$

   Performing quantifier elimination gives an estimate of the domain of attraction, since the resulting constraints on r correspond to invariant ellipsoids such that all trajectories starting in such an ellipsoid converge to the origin.

The numbers τ_i and the matrix Q in the above procedure are design parameters which can be chosen differently in order to obtain other estimates of the domain of attraction. The method proposed above cannot be used if the linearization matrices F_i do not satisfy the conditions of Lemma 2.

Example 1 *Consider the two nonlinear systems* $\dot{x} = f_i(x)$, $i = 1, 2$ *where*

$$f_1 = \begin{bmatrix} \frac{x_1}{2} + x_2 + x_2^2 \\ -x_2 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} -x_1 - x_2 \\ \frac{x_2}{3} - x_1^3 \end{bmatrix}.$$

The linearization matrices are

$$A_1 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{3} \end{bmatrix}$$

which both are unstable. However, $A_1 + A_2$ *is Hurwitz and we can switch between the two systems to stabilize the system locally according to Lemma 2. It is easy to see that* $P = I$ *solves the Lyapunov equation* (19). *To estimate the domain of attraction we utilize formula* (20) *and get* $0 < r < 1.0552$. *Hence, within a circle of radius at least* 1.055 *we can control the state to the origin.*

Lemma 2 may only be used in situations when the linearizations of the continuous dynamics contain enough structure. However, these conditions are not satisfied in general and quantifier elimination is the only tool, which we are aware of, that can handle these cases.

Example 2 *Lemma 2 cannot be used for the system* $\dot{x} = f_i(x)$, $i = 1, 2$ *where*

$$f_1 = \begin{bmatrix} -x_1 - x_1^2 \\ -x_2^3 - x_1^2 x_2^2 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} -x_1^5 - x_1^4 x_2 + x_2^2 \\ x_1^2 - x_1^4 x_2 + x_2^2, \end{bmatrix}.$$

since the linearizations do not contain enough information about the system behavior near the origin. We have to use Theorem 1 or Corollary 1 directly instead. We try with the Lyapunov candidate function $V(x) = x_1^2 + x_2^2$. *Performing quantifier elimination in formula* (12) *gives* TRUE, *which shows that the system is globally stabilizable by switching basic controllers.*

In Figure 1 we illustrate the regions where $\dot{V}_1(x) < 0$ *and* $\dot{V}_2(x) < 0$. *The union of the gray-shaded regions covers the whole state space.*

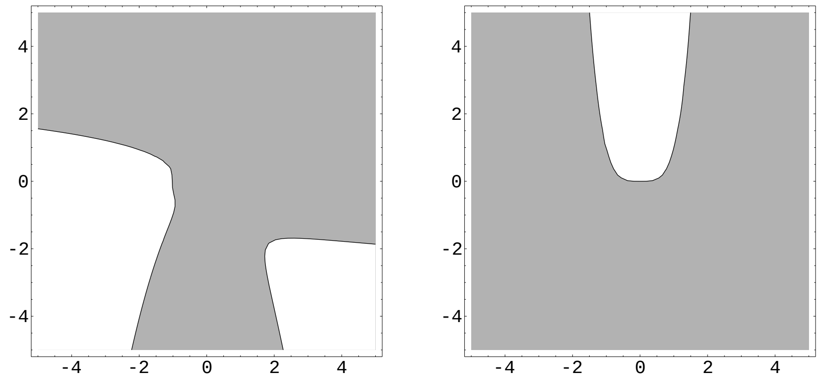

Figure 1: The regions (gray) where $\dot{V}_1(x) < 0$ (left) and $\dot{V}_2(x) < 0$ (right).

The curve $x_2 = 3x_1^2/2 - 1$ *can be shown to lie in the interior of the overlap of the regions in Figure 1 and can hence be used to switch between the systems to obtain global asymptotical stability.*

4 Exact Output Tracking

In this section we show how some of the results presented on stabilizability of switched systems can be applied to investigate the problem of exact output tracking and minimum phase properties of a class of polynomial systems, which are not affine in the control. The material in this section is based on [11].

Consider the following class of non control-affine systems

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{21}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$ and f and h are analytic vector valued functions of their arguments. The system (21) is a generalization of the nonlinear systems usually investigated in the nonlinear literature [7, 12], which is affine in the control. A good discussion on the motivation for considering the zero dynamics of the system (21) can be found in [4].

Note that the usual way of investigating systems of the form (21) is to introduce an integrator at the plant input [12], which transforms it into a control-affine system. However, the new augmented system may have the following undesirable properties according to [4]:

(i) Stabilizability of the new system with static feedback implies stabilizability of (21) by dynamic feedback.

(ii) The relative degree of the new system is higher than that of (21).

(iii) The transformation may introduce singularities.

Definition 7 *A state* $x \in \mathbb{R}^n$ *is termed an* output zeroing stationarizable state *for system (21) if it is stationarizable and* $h(x) = 0$.

Without loss of generality, it can be assumed that the origin is an output zeroing stationarizable state. We use the notation $x(t, x_0, u(\cdot))$ to denote the solution of (21) with initial value x_0.

Definition 8 *A closed set* S, $S \subset \mathbb{R}^n$, *is said to be a* viable set *of the system (21), if there exists a (continuous) feedback control law* $u = u(x)$ *defined on* S *such that there exists a solution* $x(t, x_0, u(\cdot))$ *of the system (21) which satisfies:*

$$x(t, x_0, u(\cdot)) \in S, \quad \forall x_0 \in S \quad 0 \le t < T$$

where either $T = \infty$ *or* $\lim_{t \to T^-} |x(t)| = \infty$. *Suppose that* $S_O \subset h^{-1}(0), S_O \neq \emptyset$ *is a viable set. Any control law,* $u = u(x)$, *which is defined on* S_O *is called an* output zeroing controller.

Hence, a viable set is a subset of the state space that can be made invariant by a suitable choice of state feedback.

Definition 9 *Consider the following sets*

$$\begin{aligned} L_0 &= h^{-1}(0) \\ S &= \{S_\alpha \subset L_0 : S_\alpha \textit{ viable}\} \\ M^* &= \bigcup_{S_\alpha \in S} S_\alpha \end{aligned} \tag{22}$$

If $M^* \neq \emptyset$, *a* zero dynamics *is said to exist for the system defined by (21). In other words, there exists a (continuous) feedback control law* $u(x)$ *such that for all* $x_0 \in M^*$ *it follows that* $x(t, x_0, u(\cdot)) \in M^*$ *for* $t \in [0, T], T > 0$, *i.e.,* M^* *is an invariant set of system* (21) *if* $u(x)$ *is used.*

Definitions 8 and 9 are taken from [4], where continuity of the output zeroing controllers is required. We drop this assumption in the sequel.

The following definition of minimum phase is due to [11] and differs from the usual definitions found in [7, 12]. Observe that continuity of output zeroing controllers is not required and the problem of minimum phase is that of stabilizability and not of stability of the zero dynamics.

Definition 10 *The system (21) is termed* minimum phase *at x^* if its zero dynamics is stabilizable at x^*.*

In the sequel we will assume that there exists zero dynamics and an *a priori* known output zeroing stationarizable state $x^* \in M^*$ at which we wish to investigate the minimum phase property.

Definition 11 *Suppose that there exists zero dynamics for the system (21) and an output zeroing stationarizable state x^*. Then the zero dynamics is stabilizable at x^* if there exists an output zeroing control law $u = u(x)$ with the following properties:*

(i) for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$x_0 \in M^* \cap B_\delta(x^*) \quad \Longrightarrow \quad x(t, x_0, u(x)) \in M^* \cap B_\epsilon(x^*), \quad \forall t \geq 0.$$

(ii) there exists $\Delta > 0$ such that

$$x_0 \in M^* \cap B_\Delta(x^*) \quad \Longrightarrow \quad \lim_{t\to\infty} |x(t, x_0, u(x)) - x^*| = 0$$

and we have $x(t, x_0, u(x)) \in M^*$ for all $t \geq 0$.

Here $B_d(x)$ denotes an open ball with radius d, centered at $x \in \mathbb{R}^n$. Any control law which satisfies the above given conditions is referred to as a minimum phase controller*.*

In order to analyze the minimum phase property it is very useful if we transform the system into a normal form [7]. Suppose that the system (21) has a relative degree $r \leq n$ at an output zeroing stationarizable state x^*. Then there exists a locally invertible coordinate transformation $z = \Phi(x)$ such that the system (21) is transformed into the following form [12]

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_r &= g(\eta, \xi, u) \\ \dot{\xi} &= F(\eta, \xi, u) \\ y &= z_1 \end{aligned} \tag{23}$$

where $\eta = (z_1 \ \dots \ z_r)^T$ and $\xi = (z_{r+1} \ \dots \ z_n)^T$. We say that the zero dynamics is defined by

$$\begin{aligned} 0 &= g(0, \xi, u) \\ \dot{\xi} &= F(0, \xi, u) \end{aligned} \tag{24}$$

which corresponds to Definition 9.

If we suppose that $g(0, \xi, u)$ and $F(0, \xi, u)$ are vector valued polynomial functions, we can use the result in the previous section to investigate minimum phase properties. Since u is implicitly defined by the equation $0 = g(0, \xi, u)$ there might be several solutions u for a given x, which cannot happen in the control-affine case. Analysis of minimum phase properties for non control-affine nonlinear systems which is based on linearization was considered in [11, 10].

A number of issues arise when investigating the output zeroing control laws and the stability of the corresponding zero dynamics, which are not incorporated into the known definitions of minimum phase. We will illustrate these issues in a number of examples and at the same time demonstrate how quantifier elimination can be applied to analyze the stabilizability of the zero dynamics. To be able to use quantifier elimination we specialize our study to polynomial systems.

5 Examples

In the following examples we illustrate how one can obtain discontinuous zero dynamics by using the switched controller ideas. Moreover, we illustrate how incorporating bounded controls induces "shrinking" of the region of attraction for the zero dynamics.

There may not exist continuous output zeroing controllers which yield stable zero dynamics, whereas discontinuous output zeroing controllers which achieve this may exist. This was illustrated in the paper [11]. We can also incorporate control and state constraints in the design in a straightforward manner. Hence, the approach is practical and we can construct genuine target sets which need to be reached for the exact output tracking problem.

Example 3 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= (x_2 - 5x_3 - u)(-x_2x_3 - x_2 - u) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{25}$$

The continuous output zeroing controllers are

$$u_1(x) \triangleq x_2 - 5x_3 \qquad \text{and} \qquad u_2(x) \triangleq -x_2x_3 - x_2.$$

They give the following (continuous) zero dynamics

$$\begin{cases} \dot{x}_2 = x_3 \\ \dot{x}_3 = x_2 - 5x_3 \end{cases} \qquad \begin{cases} \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_2x_3 - x_2 \end{cases}$$

It is easily seen that 0 is an unstable stationary point of both systems. Hence, by applying a continuous output zeroing controller we do not obtain stable zero dynamics. Introduce the Lyapunov function

$$V(x) = x_2^2 + x_2x_3 + x_3^2. \tag{26}$$

The derivatives of $V(x)$ *along solutions to the two systems become*

$$\dot{V}_1(x) = 2x_2x_3 + x_3^2 + x_2(x_2 - 5x_3) + 2x_3(x_2 - 5x_3)$$

and

$$\dot{V}_2(x) = 2x_2x_3 + x_3^2 + x_2(-x_2x_3 - x_2) + 2x_3(-x_2x_3 - x_2).$$

To test if it is possible to find a discontinuous (switched) control law which yields stable zero dynamics with respect to the chosen Lyapunov function, we check the decision problem (12). *We get the following formula*

$$\forall x \left[x \neq 0 \rightarrow \left[\dot{V}_1(x) < 0 \ \vee \ \dot{V}_2(x) < 0 \right] \right],$$

which can be shown to be FALSE *and hence it is impossible to prove global stability with the given Lyapunov function. However, performing quantifier elimination in*

$$\forall x \left[\left[x \neq 0 \ \wedge \ V(x) \leq 5 \right] \rightarrow \left[\dot{V}_1(x) < 0 \ \vee \ \dot{V}_2(x) < 0 \right] \right]$$

gives TRUE *and we have an estimate of the region in which there exists a minimum phase controller* $V_5 = \left\{ (x_2, x_3) \,|\, V(x) \leq 5 \right\}$. *See Figure 2 for regions where* $\dot{V}_i(x) < 0$.

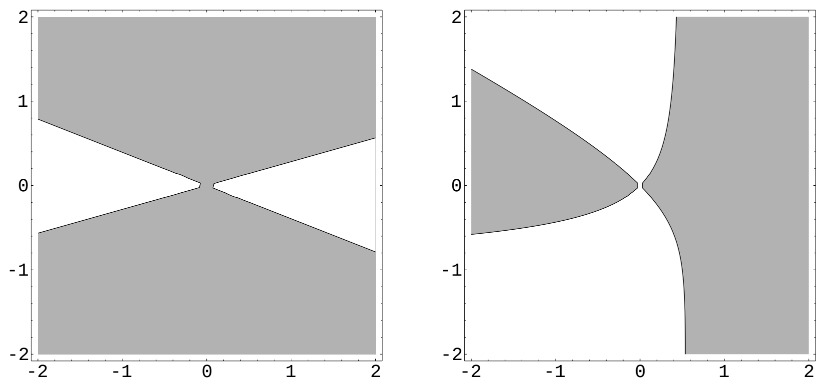

Figure 2: The regions (gray) where $\dot{V}_1(x) < 0$ (left) and $\dot{V}_2(x) < 0$ (right).

A switched controller that stabilizes the zero dynamics can be implemented using the following switching rule. Initially, choose controller according to

$$\operatorname*{argmin}_{i \in \{1,2\}} \dot{V}_i(x_0)$$

where x_0 *denotes the initial state. In the sequel, use the current controller as long as* $\dot{V}(x) < 0$, *when* $\dot{V}(x)$ *hits zero, switch to the other controller. Due to the overlap between regions defined by* $\dot{V}_1(x) < 0$ *and* $\dot{V}_2(x) < 0$, *we will always switch to a controller that makes* $\dot{V}(x) < 0$ *and the state converges to the origin.*

In Figure 3 we show some state trajectories and a time response for the zero dynamics of system (25), *when the above control law has been used.*

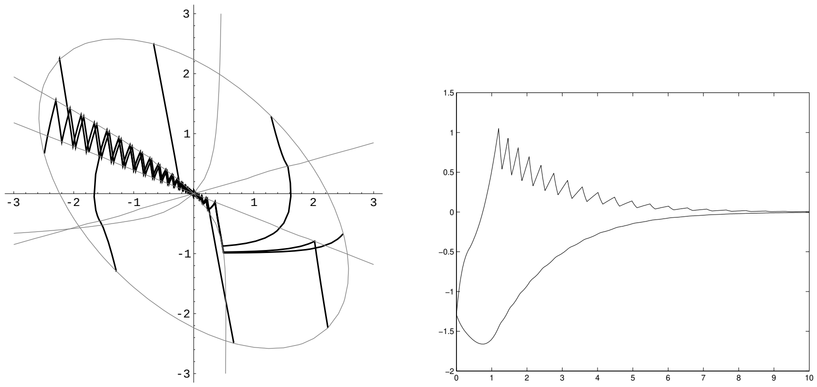

Figure 3: Left: Estimate of the region of attraction, switching surfaces, and some trajectories. Right: A time response for initial state $x_0 = (-1.3, -1.3)^T$.

It is straightforward to take control constraints into account. Suppose that we have the constraint $|u| \leq 1$. *Then by considering the following decision problem*

$$\exists r\, \forall x \Big[\big[x \neq 0 \ \wedge \ V(x) \leq r^2 \big] \ \rightarrow \\ \big[[\dot{V}_1(x) < 0 \ \wedge \ |x_2 - 5x_3| \leq 1] \ \vee \ [\dot{V}_2(x) < 0 \ \wedge \ |x_2 x_3 + x_2| \leq 1] \big] \Big]$$

we see that the region of attraction is much smaller than without control constraints. In fact, $V_{0.24} \triangleq \{ (x_2, x_3) \,|\, x_2^2 + x_2 x_3 + x_3^2 \leq 0.24 \}$ *is an estimate of the region of attraction and for only slightly larger values of* r^2 *the control constraints prevent controlled invariance of the set* V_{r^2}.

It may happen that the neither of the solutions which keep the output identically equal to zero is defined on a neighborhood of the equilibrium of interest. However, by combining the different solutions, we may achieve that the domain of definition is a neighborhood of the origin. This is a purely algebraic constraint and we are not aware of any techniques in the literature which could be used for this kind of stability analysis. The constructive power of quantifier elimination is very useful in this case.

Example 4 *The switched dynamic system considered in this example can be seen as the zero dynamics of the following system*

$$\begin{aligned} \dot{x}_1 &= (x_2 - 6x_3 - u)(x_2^2 + x_3^3 - u^2) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{27}$$

We get three different systems which correspond to the continuous output zeroing controllers

$$\begin{cases} \dot{x}_2 = x_3 \\ \dot{x}_3 = x_2 - 6x_3 \end{cases} \qquad \begin{cases} \dot{x}_2 = x_3 \\ \dot{x}_3 = \sqrt{x_2^2 + x_3^3} \end{cases} \qquad \begin{cases} \dot{x}_2 = x_3 \\ \dot{x}_3 = -\sqrt{x_2^2 + x_3^3} \end{cases}$$

Can we switch between these systems to achieve stability of the zero dynamics? Observe that the domain of definition of the state feedback control laws that give the last two systems are not the whole state space. Furthermore, the first system is unstable so the zero dynamics cannot be stabilized without switching. We will show that the zero dynamics can be stabilized by switching despite these observations. The regions where $\dot{V}_i(x) < 0$ *are shown in Figure 4.*

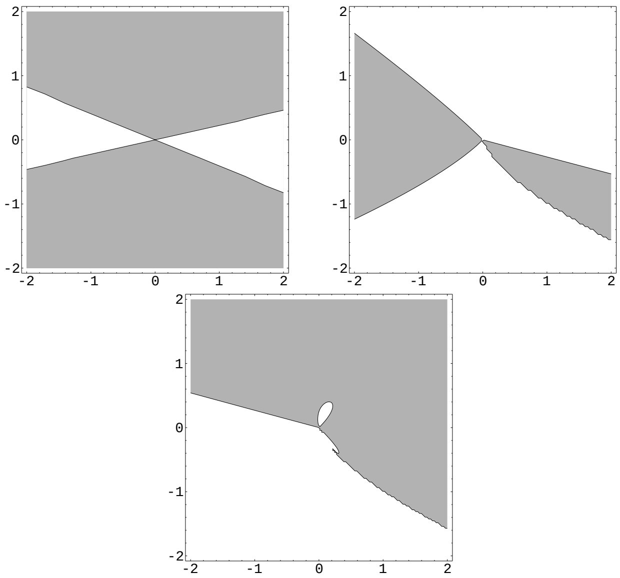

Figure 4: The regions (gray) where $\dot{V}_i(x) < 0$, $i = 1, 2, 3$.

Let $V(x) = x_2^2 + x_2x_3 + x_3^2$. *The derivatives of* $V(x)$ *along solutions to the systems are*

$$\dot{V}_i(x) = 2x_2x_3 + x_3^2 + (x_2 + 2x_3)u_i,$$

where u_i *is given by one of the solutions to* $(x_2 - 6x_3 - u)(x_2^2 + x_3^3 - u^2) = 0$. *The modification of the quantifier elimination problem* (11) *due to the implicit relation between* x *and* u *becomes*

$$\begin{aligned} \exists r\, \forall x_2\, \forall x_3\, \exists u \Big[\big[x \neq 0 \ \wedge \ V(x) \leq r^2 \big] \ \rightarrow \\ \big[2x_2x_3 + x_3^2 + (x_2 + 2x_3)(x_2 - 6x_3) < 0 \ \vee \\ 2x_2x_3 + x_3^2 + (x_2 + 2x_3)u < 0 \ \wedge \ x_2^2 + x_3^3 - u^2 = 0 \big] \Big]. \end{aligned}$$

Performing quantifier elimination we can prove that $V(x) \leq 169$ *describes a control invariant set in which the zero dynamics can be stabilized by switching.*

We investigate the example from [11] to show how quantifier elimination can be used to obtain a parameterized set of solutions for the problem – in [11] only numerical solutions were available. We will follow the procedure outlined in Section 3.

Example 5 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= a_1(x) + u + u^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= a_2(x) + u - u^3 \\ y(x) &= x_1 \end{aligned} \tag{28}$$

where $a_1(x) = -x_1 + 2.1x_2 - 2x_3$ *and* $a_2(x) = 3x_2 - 3.8x_3$. *The control objective is to design a control law which would keep the output identically equal to zero and for which the zero dynamics is stable.*

There are two output zeroing control laws

$$u_{\pm}(x) = \frac{-1 \pm \sqrt{1 - 4(2.1x_2 - 2x_3)}}{2}, \tag{29}$$

which are well defined on the set $S = \left\{ x \in \mathbb{R}^3 \mid 1 - 4(a_1(x)) \geq 0, x_1 = 0 \right\}$. *The zero dynamics is defined by* $y = x_1 \equiv 0$ *and hence we obtain*

$$\begin{aligned} \dot{x}_2 &= x_3 \\ \dot{x}_3 &= 3x_2 - 3.8x_3 + u_{\pm} - u_{\pm}^3. \end{aligned} \tag{30}$$

After linearization of (30) *at the origin with the choice* u_+ *we obtain that the linearized zero dynamics has its eigenvalues at* $\{0.41, -2.21\}$. *Similarly, the linearized zero dynamics with* u_- *has as eigenvalues* $0.10 \pm 1.09i$. *Hence, for both control laws (29) the corresponding zero dynamics is unstable.*

The Jacobians obtained by linearizing the zero dynamics (30) is denoted by F_1 *and* F_2.

We compute the Routh-Hurwitz (or Lienard-Chipart) inequalities for $\tau_1 F_1 + \tau_2 F_2$ *which are parameterized by* τ_1 *and* τ_2

$$-(3\tau_1^2 - \tau_1\tau_2 - 4\tau_2^2) > 0, \quad 9\tau_1 - \tau_2 > 0.$$

Performing quantifier elimination, we first check the existence of solution (this was verified numerically in [11])

$$\exists\tau_1\, \exists\tau_2 \left[-(3\tau_1^2 - \tau_1\tau_2 - 4\tau_2^2) > 0 \;\wedge\; 9\tau_1 - \tau_2 > 0 \;\wedge\; \tau_1 > 0 \;\wedge\; \tau_2 > 0 \right].$$

This decision problem can be shown to be TRUE *and the system can be stabilized at least locally. If we only eliminate the quantified variable* τ_1 *in above formula we obtain* $\tau_2 > 0$. *Hence, for any positive* τ_2 *there exists* τ_1 *for which the positive combination of matrices* $F(\tau_1, \tau_2) = \tau_1 F_1 + \tau_2 F_2$ *is Hurwitz.*

By fixing any pair of τ_1, τ_2 *which yields stability, the matrix* $F(\tau_1, \tau_2)$ *satisfies a Lyapunov matrix equation. That is, for any matrix* $Q = Q^T \in \mathbb{R}^{n \times n}$ *with* $Q \succ 0$ *there exists a matrix* $P = P^T \in \mathbb{R}^{n \times n}$ *with* $P \succ 0$ *which is the solution of*

$$F^T(\tau_1, \tau_2)\, P + P\, F(\tau_1, \tau_2) = -Q.$$

Note that Q *introduces another degree of freedom and can be regarded as a design parameter. The following* P *is a solution of the above matrix equation*

$$P = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

With this P *we estimate the domain of attraction of the zero dynamics, which is the target set we want to reach if we want exact tracking of constant outputs. By using quantifier elimination we obtain that the ellipsoid defined by* $x^T P x < 0.057735$ *is a domain of attraction for the zero dynamics. In fact, the plane* $1 - 4a_1(x) = 0$ *which is the boundary of the domain of definition of the switched control laws, is a tangent to this ellipsoid.*

The zero dynamics is usually described by *nonlinear* differential equations and yet this fact is not incorporated sufficiently into the definition of minimum phase. For instance if the zero dynamics has a globally stable limit cycle, the system would be termed non-minimum phase despite that the behavior of the systems when tracking constant outputs may be satisfactory. In this sense the known definition of minimum phase is often misleading. This was tried to overcome in [2] where global stability of an invariant set was suggested as the definition of minimum phase. We believe that the local equilibrium stability and global set stability definitions are just two ends of a wide spectrum of possible situations.

In the following example we show how to use quantifier elimination to estimate the region of attraction of an invariant set for a given zero dynamics.

Example 6 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= (x_2 x_3^2 - x_2^2 - u)(x_2^2 - x_2 - u)(x_2 - 4x_3 - u) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \\ y &= x_1 \end{aligned} \tag{31}$$

The question of invariance of a given set can be formulated as follows: Is it possible to choose the control such that the solution trajectory tangent always points inwards along the boundary of the given set, i.e., $V_x(x) f(x, K_i(x)) < 0$. *Using the Lyapunov function* $V(x) = x_2^2 + x_2 x_3 + x_3^2$, *we can show that the following formula is* TRUE

$$\begin{aligned} \forall x_2 \forall x_3 \Big[\tfrac{1}{2} \le V(x) \le 61 \to & \\ & \big[2x_2x_3 + x_3^2 + (x_2 + 2x_3)(x_2x_3^2 - x_2^2) < 0 \ \vee \\ & \ 2x_2x_3 + x_3^2 + (x_2 + 2x_3)(x_2^2 - x_2) < 0 \ \vee \\ & \qquad 2x_2x_3 + x_3^2 + (x_2 + 2x_3)(x_2 - 4x_3) < 0 \big] \Big] \end{aligned}$$

which means that the invariant set $V(x) \le 1/2$ *has a domain of attraction defined by* $V(x) \le 61$. *If we change* $V(x) \ge 1/2$ *to* $V(x) \ne 0$ *the formula is* FALSE. *Further analysis is needed to decide how the state behaves inside the ellipsoid* $V(x) \le 1/2$ *(asymptotically stable, limit cycles, etc).*

Some of the issues treated here were raised in [11] but we show that for a large class of systems, quantifier elimination provides a tool to test these conditions.

The use of switched output zeroing controllers allow for more flexibility when control and state constraints have to be taken into account. If the goal is to exactly track constant outputs, then fixing the equilibrium around which we wish our zero dynamics to be stable, does not seem to be appropriate. Indeed, it may happen that the zero dynamics system has several different equilibria with (perhaps) disjoint regions of attractions. In order to exactly track the output and have bounded states and control, reaching any of the basins of attraction would satisfy the control objective. Quantifier elimination could be used for such computations.

6 Conclusions

We have investigated the application of quantifier elimination to the analysis of stabilizability of polynomial systems by switching between a number of controllers. We have shown how to check if a polynomial system is stabilizable by switching but also how to estimate the region of attraction in such cases.

In exact output tracking, the zero dynamics plays an important role. For control-affine systems the zero dynamics is uniquely determined and stability of the zero dynamics or, equivalently, the concept of minimum phase, is usually used as a feasibility condition for exact output tracking problems. For more general nonlinear systems the zero dynamics or the output zeroing controller are no longer unique. We argue that stabilizability of the zero dynamics is a more appropriate condition than just stability. In a number of examples we show how to stabilize the zero dynamics of different systems by switching between different output zeroing controllers, where no continuous stabilizing controller exists. Quantifier elimination is used to carry out the necessary computations, where we also can take control and state constraints into account in a direct manner.

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