Solving LMI and BMI Problems by Quantifier Elimination

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1 Introduction

Several interesting control system design and analysis problems can be reduced to quantifier elimination (QE) problems as shown in [3],[7],[1],[11],[12]. The first attempt by Anderson *et al.*[3] was made in 1970's, but at that time the algorithm of QE was very intricate and no appropriate software was available. However, recently some improved algorithms have been developed by several authors (see [5],[6],[19]) and implemented on computers (see [10],[15],[16]). By virtue of the considerable developments of both algorithms and software in QE methods, we explore the application of the QE theory to control problems of great practical interest.

Many control problems and design specifications are described by using matrix inequalities. Here we focus on LMI and BMI problems; When $A \in \mathbf{R}^{n\times n}$ is *(semi) positive definite,* we denote it by $A >_d 0$ ($\geq_d 0$). A *linear matrix inequality (LMI)* is a matrix inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \tag{1}$$

where $x \in \mathbf{R}^m$ is the variable vector and $F_i = F_i^T \in \mathbf{R}^{n\times n}$ for $i = 0, \cdots, m$. And a *bilinear matrix inequality (BMI)* is of the form

$$G(x,y) = G_{0,0} + \sum_{i=1}^{n_x} x_i G_{i,0} + \sum_{j=1}^{n_y} y_j G_{0,j} + \sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j G_{i,j} \tag{2}$$

where $x \in \mathbf{R}^{n_x}$, $y \in \mathbf{R}^{n_y}$, $G_{i,j} = G_{i,j}^T \in \mathbf{R}^{n\times n}$ for $i \in \{0, \cdots, n_x\}$, $j \in \{0, \cdots, n_y\}$.

A number of important problems can be are reduced to the optimization problems with LMI or BMI constraints. The problem of minimizing a linear objective function in $x \in \mathbf{R}^m$ subject to LMI constraint,

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & F(x) \geq_d 0 \end{array} \tag{3}$$

where $c \in \mathbf{R}^m$, is called *semidefinite programming (SDP)*. SDP problem is solved numerically as convex optimization problems (see [4],[20]). But when we consider the real parametric uncertainties the problems are not always convex (often become non-convex) and most of such methods does not work.

Moreover, we also consider *extended SDP (ESDP) i.e.* the extension of the SDP obtained by replacing LMI constraint by BMI

$$\begin{array}{ll} \textit{minimize} & c^T x + b^T y \\ \textit{subject to} & G(x, y) \geq_d 0 \end{array} \tag{4}$$

where $c \in \mathbf{R}^{n_x}$, $b \in \mathbf{R}^{n_y}$. Though BMIs includes much wider variety of control problems than LMIs, BMI problems are non-convex and, in general, hard to find the solution numerically.

2 Quantifier Elimination

Many mathematical and industrial problems can be translated to formulas consisting of polynomial equations, inequalities, quantifiers $(\forall, \exists)$ and Boolean operators $(\wedge, \vee, \neg, \rightarrow, etc)$. Such formulas construct sentences in the so-called *first-order theory* of real closed fields and are called *first-order formulas*.

Let $f_i(X, U) \in \mathbf{Q}[X, U]$, $i = 1, 2, \cdots, t$, where $\mathbf{Q}$ is the fields of rational numbers, $X = (x_1, \cdots, x_n) \in \mathbf{R}^n$ a vector of quantified variables, and $U = (u_1, \cdots, u_m) \in \mathbf{R}^m$ a vector of unquantified parameter variables. Let $F_i = f_i(X, U) \;\square_i\; 0$, where $\square_i \in \{=, \geq, >, \neq\}$, for $i = 1, \cdots, s$, $Q_j \in \{\forall, \exists\}$, and X_j a block of q_j quantified variables for $j = 1, \cdots, s$. In general, quantified formula φ is given

$$\varphi = ({}^{Q_1}X_1 \cdots {}^{Q_s}X_s)\; G(F_1, \cdots, F_t) \tag{5}$$

where $G(F_1, \cdots, F_t)$ is a quantifier-free (qf) Boolean formula.

QE procedure is an algorithm to compute equivalent qf formula for a given first-order formula. If all variables are quantified, *i.e.* $m = 0$, QE procedure decides whether the given formula (5) is *true* or *false*. This problem is called *decision problem*. When there are some unquantified variables U, QE procedure find a qf formula $\varphi(U)$ describing the range of possible U where $\varphi(U)$ is true. If there is no such range QE outputs false. This problem is called *general quantifier elimination problem*.

The history of the algorithms QE begins with *Tarski-Seidenberg decision procedure* in 1950's ([17],[14]). But this is intricate and far from feasible. In 1975, Collins presented a more efficient general purpose QE algorithm based on Cylindrical Algebraic Decomposition (CAD) [5]. The algorithm has improved by Collins and Hong in [6] and was implemented on SACLIB as "QEPCAD" by Hong. Weispfenning also presented a very efficient QE algorithm based on test terms [18],[19]. Though there is some degree restriction of a quantified variable in input formulas for this approach, it seems very practical. Implementation of the method was done on Reduce as "REDLOG" and *Risa/Asir* [1] [13] by Sturm [15], [16]. Here we use both QE implementation on *Risa/Asir* and REDLOG.

3 Existing methods for LMI & BMI problems

As for LMI problems, we can obtain the numerical solution by applying the efficient algorithm of convex optimization which based on the interior point method to SDP problems, if we can not obtain the analytic solution. As Software tools [8] for solving LMI based on the convex optimization methods are widely spread, LMI approach becomes one of the powerful design methods [4].

In the case of BMI problems, since BMI is a non-convex constraint, we can not utilize the convex optimization methods. This fact makes it difficult to solve BMI problems. For a given BMI, if we can derive a LMI which is equivalent to the BMI, the BMI problem is solved as the LMI problem by convex optimization method. There are many successful results in this direction. But there exists many important control and design problems described by BMI which can not be reduced to LMI problems. The method using branch & bound is proposed [9] for such problems. However, there remains some issues in the branch & bound approach.

[1] *Risa/Asir* is a computer algebra system developed at Fujitsu Labs Ltd. FTP:*endeavor.fujitsu.co.jp:/pub/isis/asir*

Due to the non-convexity of BMI, it is difficult to obtain the answer in the whole space. So we compute the answer in a bounded subspace. It is not easy to find the "meaningful" bounded subspace. Moreover, it is difficult to determine exactly infeasibility by branch & bound approach

Therefore, it is desirable to develop the methods which also work for non-convex case and parametric case. The QE method does not utilize any information about possible convexity of the problem and hence does not suffer from this drawback. So here we present a new symbolic method based on QE for the SDP and ESDP problems and show some experiment by using existing QE package to demonstrate the capability of the method.

4 Reducing SDP to QE problems

In this section we consider the method to solve BMI problems by using QE [2]. This method is applicable to not only BMI but also general matrix inequality (Polynomial Matrix Inequality; PMI). Here we denote a polynomial matrix inequality by $M(X) >_d (or \geq_d) 0$, where $X = \{x_1, \cdots, x_n\}$ and we use $>_d$ $(\geq_d)$ for definiteness of matrices in order to distinguish ordinary inequality sign.

Determining (semi)definiteness for a real symmetric matrix is achieved without computing eigenvalues by using the following well-known as Sylvester's theorem; For a matrix $M \in \mathbf{R}^{n\times n}$, we denote by

$$M \begin{pmatrix} i_1 \; i_2 \; \cdots \; i_r \\ j_1 \; j_2 \; \cdots \; j_r \end{pmatrix}$$

the $r \times r$ submatrix of M which consists of (i_k, j_l)-entries of M, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq n$.

Theorem 1 *(*Sylvester's criterion*) Let $M = (a_{ij}) \in \mathbf{C}^{n\times n}$ be a Hermitian matrix. Then*

(i) M is positive semi-definite if and only if all principal minors of M are non negative i.e.

$$det\ M \begin{pmatrix} i_1 \; i_2 \; \cdots \; i_r \\ i_1 \; i_2 \; \cdots \; i_r \end{pmatrix} \geq 0, \tag{6}$$

for $1 \leq i_1 < i_2 < \cdots < i_r \leq n,\ r = 1, 2, \cdots, n.$

(ii) M is positive definite if and only if all leading principal minors of M are positive i.e.

$$det\ M \begin{pmatrix} 1 & 2 & \cdots & r \\ 1 & 2 & \cdots & r \end{pmatrix} > 0 \quad for\ r = 1, 2, \cdots, n. \tag{7}$$

This is the key to reduce SDP problems to QE problems. By Theorem 1 (i), $M(x) \geq_d 0$ can be reduced to the formula which is the conjunction of $2^n - 1 (\equiv \sum_{r=1}^{n} {}_nC_r)$ inequalities.

Remark 1 *By this theorem, Positive-Definite Programming can be also resolved in the same manner as SDP problems if we use (ii).*

5 Optimization by QE and SDP

By Sylvester's criterion semidefinite constraints are reduced to a conjunction of inequalities and the SDP problems are reduced to the ordinary nonlinear programming problems.

In [21] Weispfenning showed that the optimization problem, in particular linear programming problems, can be solved successfully by using his highly improved QE algorithm. We utilize his algorithm for a nonlinear programming problem derived from the SDP problem as above. This explains how we solve the SDP problems by using QE.

Here we show the brief sketch of his method to solve optimization problems given by a Boolean system $\psi(X, U)$ consisting of equations and inequalities and an objective function $h(X, U)$; First introduce a new indeterminate z assigned to the object function h. Consider the new Boolean system $\psi' = \psi \ \wedge (z - h \geq 0)$. Then the problem minimizing h subject to $\psi(X, U)$ is formulated as first-order formula $\varphi = \ {}^{\exists}x_1 \cdots \ {}^{\exists}x_n(\psi')$. Next eliminate all quantified variables $x_1, \cdots, x_n$ to have the resulting qf formula φ' in z and U. After that we specialize parameters U of φ' by an appropriate real values, then φ' gives a finite union M of intervals for z, which shows a possible range of z. If M is empty, ψ is unsolvable (*i.e.* infeasible); if M is unbounded from below, h has no minimum w.r.t. ψ; if $m \in M$ is a lowest endpoint of M, then m is the minimum value of z w.r.t. ψ (for details [21]). As for the coordinates of a minimum point for z can be obtained by *back-substitution* from the test terms used during the elimination if we use test term approach (see [21]).

6 Solving BMI by QE

Algorithm 1 (Feasible problems) "Find X which satisfies $M(X) >_d$ *(or* $\geq_d$*)* 0"

1. *Transform* $M(X) >_d$ *(or* $\geq_d$*)*0 *to an equivalent formula* $\varphi(X)$ *by using Sylvester's theorem.*
2. *Construct the first-order formula* $\psi = \exists X(\varphi(X))$.
3. *Perform QE for* ψ.

QE outputs true or false. In the case of true QE returns one feasible solution. false means there are no feasible solutions.

Example 1 *"Does BMI* $B(x_1, x_2) <_d 0$ *have any feasible solutions in* $(x_1, x_2) \in [-\frac{1}{2}, 2] \times [-3, 7]$ *? [9]" where*

$$
\begin{aligned}
B(x_1, x_2) = &\begin{bmatrix} -10 & -\frac{1}{2} & -2 \\ -\frac{1}{2} & \frac{9}{2} & 0 \\ -2 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} -\frac{9}{5} & -\frac{1}{10} & -\frac{2}{5} \\ -\frac{1}{10} & \frac{6}{5} & -1 \\ -\frac{2}{5} & -1 & 0 \end{bmatrix} \\
&+x_2 \begin{bmatrix} 9 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} + x_1 x_2 \begin{bmatrix} 0 & 0 & 2 \\ 0 & -\frac{11}{2} & 3 \\ 2 & 3 & 0 \end{bmatrix}.
\end{aligned}
$$

This problem is equivalent to the following QE problem:

$$\exists x_1 \exists x_2 (A_1 \wedge A_2 \wedge A_3 \wedge -\frac{1}{2} \leq x_1 \leq 2 \wedge -3 \leq x_2 \leq 7)$$

where

$$A_1 = -45x_1 + 9x_2 + 50 > 0,$$

$$
\begin{aligned}
A_2 = &(-4950x_2 - 25)x_1^2 + (990x_2^2 + 6590x_2 + 4100)x_1 - 217x_2^2 \\
&-2020x_2 - 4525 > 0,
\end{aligned}
$$

$$
\begin{aligned}
A_3 = &(-11000x_2^3 + 37500x_2^2 - 102750x_2 + 40375)x_1^3 + (-700x_2^3 \\
&-15850x_2^2 + 141250x_2 - 27500)x_1^2 + (3680x_2^3 + 12935x_2^2 \\
&-71900x_2 - 19625)x_1 - 764x_2^3 - 3280x_2^2 + 7000x_2 + 9000 > 0.
\end{aligned}
$$

After performing QE for this, we have "true". In the case of $(x_1, x_2) \in \mathbf{R}^2$, *after performing QE for* $\exists x_1 \exists x_2 (A_1 \wedge A_2 \wedge A_3)$ *we also have true.*

Algorithm 2 (linear objective minimization problems) *"minimize 'c$^T X$ subject to $M(X) >_d$ (or $\geq_d$)0.*

1. *Transform $M(X) >_d$ (or $\geq_d$)0 to an equivalent formula $\varphi(X)$ by using Sylvester's theorem.*
2. *Assign the new variable z to the objective function; $z - c^T X \geq 0$.*
3. *Construct the first-order formula $\psi = \exists X \ (\varphi(X) \ \wedge \ z - c^T X \geq 0)$.*
4. *Perform QE for ψ.*

We consider some examples (which are modified one taken from [20]), and apply the above mentioned method to them in order to demonstrate the potential of QE approach to SDP problems.

• **Non-Parametric:** *Minimize $x_1 + x_2$ subject to*

$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1+1 \end{bmatrix} \geq_d 0. \tag{8}$$

The semidefinite constraint (8) is reduced to a Boolean system of inequalities constraints by using Sylvester's criterion;

$\psi = (1 \geq 0) \wedge (x_2 \geq 0) \wedge (x_1 + 1 \geq 0) \wedge (x_2 - x_1^2 \geq 0) \wedge (x_2 x_1 + x_2) \geq 0) \wedge (x_1 + 1 \geq 0) \wedge (-x_1^3 - x_1^2 + x_2 x_1 + x_2 \geq 0)$.

Assign new slack variable z to objective function $x_1 + x_2$ and let $\psi' = \psi \wedge (z - x_1 - x_2 \geq 0)$. Then the problem is formulated as a first-order formula $\varphi = \ ^{\exists}x_1 \ ^{\exists}x_2 \ (\psi')$. After QE we have an equivalent qf formula describing the range of objective function;

$$4z + 1 \geq 0 \tag{9}$$

• **Parametric(1):** *Minimize $ax_1 + x_2$ subject to (8) with a parameter a.*

Let $\psi'_{P_1} = \psi \wedge (z - ax_1 - x_2 \geq 0)$. The problem is formulated as $\varphi_{P_1} = \ ^{\exists}x_1 \ ^{\exists}x_2 \ (\psi'_{P_1})$. After QE we have an equivalent qf formula describing the range of objective function z with a parameter a;

$(a+z-1 \geq 0) \vee (z \geq 0) \vee ((a^3 - 2a^2 z + a^2 + az^2 - az \leq 0) \wedge (a^2 - az \geq 0) \wedge (a - z + 1 \leq 0) \wedge (a = 0)) \vee ((a^3 - a^2 + 3az - 2z \leq 0) \wedge (a^2 + 4z \geq 0) \wedge (a^2 + 2z \geq 0) \wedge (a - 2 \leq 0)) \vee ((a^2 + 4z \geq 0) \wedge (a - 2 < 0))$.

If we substitute a parameter a with 1 and simplify the result, we have same result as (9).

• **Restricted region:** *Minimize x_1+x_2 subject to (8) in restricted domains D_1, D_2 respectively, where (i) $D_1 = \{x_1, x_2 \in \mathbf{R} \mid 0 \le x_1 \le 3, 0 \le x_2 \le 5\}$ and (ii) $D_2 = \{x_1, x_2 \in \mathbf{R} \mid 0 \le x_1 \le 10, 5 \le x_2 \le 10\}$.*

(i) In this case the problem is formulated as

$$\varphi_{D_1} = {}^{\exists}x_1\ {}^{\exists}x_2(\psi' \wedge (0 \le x_1 \le 3 \wedge 0 \le x_2 \le 5))$$

After QE we have $4z+1 \ge 0$.

(ii) In this case the problem is formulated as

$$\varphi_{D_2} = {}^{\exists}x_1\ {}^{\exists}x_2(\psi' \wedge (0 \le x_1 \le 10 \wedge 5 \le x_2 \le 10))$$

After QE we have

$(z-4 \ge 0) \vee ((z^2-10z+20 \le 0) \wedge (z^2-20z+90 \ge 0) \wedge (4z+1 \ge 0) \wedge (z \le 0)) \vee ((z^2-10z+20 \le 0) \wedge (z^2-20z+90 \ge 0) \wedge (4z+1 \ge 0) \wedge (z < 0))$.

• **Parametric(2):** *Minimize x_1+x_2 subject to*

$$\begin{bmatrix} s & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1+1 \end{bmatrix} \ge_d 0. \tag{10}$$

where s is a parameter.

The semidefinite constraint (10) is reduced to

$\psi_P = (s \ge 0) \wedge (x_2 \ge 0) \wedge (x_1+1 \ge 0) \wedge (sx_2 - x_1^2 \ge 0) \wedge (x_2x_1 + x_2) \ge 0) \wedge (s(x_1+1) \ge 0) \wedge ((x_2x_1+x_2)s - x_1^3 - x_1^2 \ge 0)$.

Let $\psi'_{P_2} = \psi_P \wedge (z - x_1 - x_2 \ge 0)$. Then the problem is formulated as a first-order formula $\varphi_{P_2} = {}^{\exists}x_1\ {}^{\exists}x_2\ (\psi'_{P_2})$. After QE we obtain

$(sz+s-1 \ge 0 \wedge s > 0 \wedge z+1 \ge 0) \vee (sz+s-1 \ge 0 \wedge s \ge 0 \wedge z+1 \ge 0) \vee (sz+s \ge 0 \wedge s \ge 0 \wedge z = 0) \vee (s^2+4sz \ge 0 \wedge ((s^2-2s \le 0 \wedge sz+s-1 \le 0 \wedge s-2 \le 0 \wedge s \ge 0 \wedge (s^2+3sz-s-2z \le 0 \vee (s+z-1 \le 0 \wedge (sz+s-1 = 0 \vee z = 0))) \wedge (s+2z \ge 0 \vee z = 0)) \vee (s+2z \ge 0 \wedge s \ge 0 \wedge (s^2-2s \le 0 \vee sz+s-1 \ge 0 \vee s = 0) \wedge (s^2-2s \le 0 \vee s > 0) \wedge ((s^2+3sz-s-2z \le 0 \wedge (sz+s-1 \le 0 \vee z = 0)) \vee (s+z-1 \ge 0 \wedge (sz+s-1 \ge 0 \vee z = 0))) \wedge (sz+s-1 \ge 0 \vee s-2 \le 0)))) \vee (s \ge 0 \wedge z \ge 0) \vee (s^2-2s \le 0 \wedge s+4z \ge$

$0 \wedge s-2 \leq 0 \wedge s>0) \vee (s^2 z+s^2-s \geq 0 \wedge s>0) \vee (s^2-2s \leq 0 \wedge s+4z \geq 0 \wedge s-2 < 0 \wedge s>0) \vee (s>0 \wedge z \geq 0) \vee (s>0 \wedge (s^2 z+s^2-s>0 \vee (s^2-2s<0 \wedge sz+s-1=0)))$.

If we substitute the parameter s with 1 and simplify the result, we have same result as (9).

- **Parameter uncertainty:** *Minimize x_1+x_2 subject to (10) within the regions of a parameter s (i) $-5 \leq s \leq -1$, (ii) $-10 \leq s \leq 0$, respectively.*

For (i),(ii), the problem is formulated as a first-order formula; $\varphi_{(i)} = {}^{\exists}x_1\ {}^{\exists}x_2(\psi'_{P_2} \wedge (-5 \leq s \leq -1))$, $\varphi_{(ii)} = {}^{\exists}x_1\ {}^{\exists}x_2(\psi'_{P_2} \wedge (-10 \leq s \leq 0))$, respectively. After QE we obtain "false" for (i) and for (ii)

$(sz+s-1 \geq 0 \wedge s=0 \wedge z+1 \geq 0) \vee (sz+s \geq 0 \wedge s=0 \wedge z=0) \vee (s^2+4sz \geq 0 \wedge ((s^2-2s \leq 0 \wedge sz+s-1 \leq 0 \wedge s=0 \wedge (s^2+3sz-s-2z \leq 0 \vee (s+z-1 \leq 0 \wedge (sz+s-1=0 \vee z=0))) \wedge (s+2z \geq 0 \vee z=0)) \vee (s^2-2s \leq 0 \wedge s+2z \geq 0 \wedge s=0 \wedge ((s^2+3sz-s-2z \leq 0 \wedge (sz+s-1 \leq 0 \vee z=0)) \vee (s+z-1 \geq 0 \wedge (sz+s-1 \geq 0 \vee z=0)))))) \vee (s=0 \wedge z \geq 0)$.

7 Conclusion

We showed that the SDP problems, which is one of the greatly important problems in LMI problems, are reduced to quantifier elimination problems by Sylvester's criterion and presented the concrete procedure to solve them by using QE.

The efficiency of the method proposed here depends on deeply that of QE procedure. Due to the great theoretical and practical complexity of QE for the elementary theory of the real closed field, currently the method is practically applicable to moderate size problems. However, compared with existing (numerical) methods for SDP and ESDP problems the QE based method has many merits as follows;

- Non-convex case can be dealt with,
- Parametric case can be dealt with,
- Easy to handle the domain restricted case and parameter uncertainty,
- Produce exact answer and check the infeasibility exactly.

Furthermore, it is important and necessary to examine the practically effective usage of QE for not only LMI & BMI but also other control problems hereafter.

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