Accuracy Analysis of Hybrid Rational Interpolation

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Abstract

Hybrid Rational Function Approximation (HRFA) is one of the most important applications of approximate-GCD algorithms. Classical rational interpolation may not yield useful approximation of continuous function by the presence of poles over the approximation range. In HRFA, the poles are removed by computing approximate-GCD of the numerator and the denominator polynomials of the interpolated rational function. In this paper, a method of how to estimate the numerical error of rational approximation obtained by HRFA is proposed by using the approximate-GCD proposed by Hribernig and Stetter. A theorem for the error estimation is established.

1 Introduction

Let $x_0 < x_1 < \cdots < x_{m+n}$ be $m+n+1$ distinct points, and let $f_i := f(x_i), i = 0, \cdots, m+n$, where $f(x)$ is an unknown function. They are changed to a set of data

$$D = \{(x_0, f_0),\ (x_1, f_1),\ \cdots, (x_{m+n}, f_{m+n})\}. \tag{1}$$

The classical algebraic problem of rational interpolation is to compute a pair of polynomials $P(x)$ and $Q(x)$ satisfying the relations deg $P(x) \le m$, deg $Q(x) \le n$,

$$\frac{P(x_i)}{Q(x_i)} = f_i,\ i = 0, \cdots, m+n, \tag{2}$$

for a given pair of nonnegative integers m and n. The rational interpolation (2) is computed by several methods, e. g. linearized equations, Thiele interpolating continued fraction and so on. Here the linearized equations are expressed as following.

$$P(x_i) - f_i \cdot Q(x_i) = 0,\ i = 0, \cdots, m+n. \tag{3}$$

If we put $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + 1$, then we can compute the values a_i and b_j for $i = 0, \cdots, m$, $j = 1, \cdots, n$ by solving the equations. However, there are two difficulties in which a rational function satisfying linearized equations would not be considered as a solution of the problem above:

- There may arise the rational interpolation of the form $\frac{0}{0}$ at x_i, i.e. *unattainable point.*
- There may be poles between the points of interpolation.

In the latter case, if a function $f(x)$ is continuous in $[x_0, x_{m+n}]$, an interpolated rational function $P(x)/Q(x)$ of f_i becomes a poor approximation at the poles and near of the poles. Braess proved a condition showing that the interpolated rational function approximation of $f(x)$ has no poles [1]. However, there is no discussion on the condition for a data set defined as D.

To avoid the difficulties, several rational interpolations have been proposed. One of them is called linear rational interpolation by Berrut and Mittelmann [2]. This interpolation has no unattainable points and no poles in the interval of the interpolation $[x_0, x_{m+n}]$. However, the degree of the approximation will become very high to obtain better approximation of $f(x)$. Thus it is difficult to use the approximation in other hybrid applications, e.g. hybrid integral.

If $Q(x)$ has zeros in $[x_0, x_{m+n}]$, i.e. a rational interpolation has poles, and $P(x)$ has close zeros to the zeros of $Q(x)$, hybrid rational function approximation (HRFA) [4] based on symbolic-numeric hybrid computation is available to remove the poles in $[x_0, x_{m+n}]$. However, if unattainable points occur in the data set D, HRFA becomes ill-conditioned. Hereafter is a detailed discussion of HRFA.

In HRFA, the approximate-GCD by Sasaki and Noda [11] is used to remove the undesired poles in $[x_0, x_{m+n}]$. For the computation of approximate-GCD, the quantity $p(x_i) - f_i \cdot q(x_i)$ in general is different from zero (even if it may be closed to zero), where $p(x)$ and $q(x)$ are the numerator and denominator polynomials of rational function obtained by HRFA. We must estimate the accuracy δ of the hybrid rational interpolation $r_{k,l}(x)$ as

$$|f(x_i) - r_{k,l}(x_i)| = \left| f(x_i) - \frac{p_k(x)}{q_l(x)} \right| \leq \delta, \ i = 0, \cdots, m+n. \tag{4}$$

where degrees of $p_k(x)$ and $q_l(x)$ are k and l respectively and $k \leq m, l \leq n$. δ is small positive real number.

In this paper, we propose a method of the error estimation of HRFA. For this purpose, the approximate-GCD proposed by Hribernig and Stetter is used. HRFA and the approximate-GCD are briefly summarized in **2** and **3**, respectively. In **4**, We show a theorem of the accuracy of HRFA using the approximate-GCD. A symbolic-numeric hybrid example is shown in **5**. The result satisfies the theorem.

2 Hybrid Rational Function Approximation

Computations of linearized equations (3) in the presence of roundoff errors may lead to the following interesting phenomenon. Suppose that the unknown function $f(x)$ is a rational function which is continuous in $[x_0, x_{m+n}]$. Let the actual degrees of the numerator and the denominator polynomials of $f(x)$ be $m-d$ and $n-d$, respectively, for a positive d. That is $f(x) = \bar{r}_{m-d,n-d}(x) = \bar{p}_{m-d}(x)/\bar{q}_{n-d}(x)$. Here, we consider rational function $\bar{r}_{m,n}(x)$ interpolating D such as $f_i = f(x_i),\ i = 0, \cdots, m+n$. There are a lot of rational functions passing the data set D but have different degrees. There are any polynomials $\bar{g}(x)$ whose degree is d such as $\bar{r}_{m,n}(x) = (\bar{g}(x)\bar{p}_{m-d}(x))/(\bar{g}(x)\bar{q}_{n-d}(x))$. Thus, linearized equations (3) have no unique solution. However, in the floating number computations, because of the presence of roundoff errors the numerical solution will produce the pair of polynomials $P(x)$ and $Q(x)$ of degrees m and n, respectively. The numerical rational interpolation $P(x)/Q(x)$ should approximate one of $\bar{r}_{m,n}$. That is $P(x)/Q(x) = (g_p(x)p_{m-d}(x))/(g_q(x)q_{n-d}(x))$. In algebraic computations we can remove $\bar{g}(x)$ from rational function $\bar{r}_{m,n}(x)$ by GCD computations. However in floating computations the coefficients of $P(x)/Q(x)$ are slightly different from $\bar{r}_{m,n}(x)$. Thus, polynomials $g_p(x)$ and $g_q(x)$ remain in the numerical rational interpolation. Furthermore, it may occur that for some z we have simultaneously $Q(z) = 0,\ P(z) \neq 0$, and then the computed ratio $P(x)/Q(x)$ will be a very poor approximation to $f(x)$ at z and near z. In this case, a natural way to improve the approximation is to compute $g(x)$, an approximate-GCD of $P(x)$ and $Q(x)$, then compute the symbolic quotients $p(x) = P(x)/g(x)$ and $q(x) = Q(x)/g(x)$, and finally output $p(x)/q(x)$ as an improved hybrid rational interpolation to $f(x)$. We call these procedures *Hybrid Rational Function Approximation* and show the algorithm as follows:

Algorithm 2.1 (HRFA)
Input: *data set D and parameter ϵ*
Output: *hybrid rational interpolation $p(x)/q(x)$*
Method:

1. *Compute rational interpolation $P(x)/Q(x)$ to interpolate D.*
2. *Obtain the approximate-GCD $g(x)$ of $P(x)$ and $Q(x)$ using the algorithm by Sasaki and Noda.*
3. *$p(x) = quo(P(x)/g(x))$ and $q(x) = quo(Q(x)/g(x))$, where quo is a quotient obtained by polynomial division.*

Any algorithms for (non-unique) approximate-GCD and for polynomial division are available. Several methods of calculating approximate-GCDs have been proposed other than Schönhage [7], Sasaki and Noda [11]. They use different norms for the definition of approximate-GCDs, i.e. L_1 norm (c.f. Pan[9], Hribernig and Stetter[8]), L_2 norm (c.f. Corless, Gianni, Trager and Watt[6], Karmarkar and Lakshman[5]) or norms estimated in a domain (c.f. Sederberg and Chang[10]). For further discussions to establish the error estimate of HRFA, we use the approximate-GCD proposed by Hribernig and Stetter.

3 Approximate-GCD by Hribernig and Stetter

The approximate-GCD, which is called *near-GCD* in [8], of two polynomials $f_1, f_2 \in C[x]$, where $C[x]$ is a complex number coefficient polynomial, is stated as the following concept.

Definition 3.1 (Hribernig and Stetter) *At the accuracy level α, two polynomials $\tilde{f}_i \in C[x]$, $i = 1, 2$, possess a near-GCD $\tilde{g}$ if there exist polynomials $f_i^* \in C[x]$, $i = 1, 2$, which satisfy*

$$GCD(f_1^*, f_2^*) = \tilde{g}, \text{ and } \|\tilde{f}_i - f_i^*\| \leq \alpha, i = 1, 2, \tag{5}$$

and $deg(\tilde{g})$ is the highest possible degree for (5). Equivalently, a near-GCD $\tilde{g}$ of $\tilde{f}_1$ and $\tilde{f}_2$ satisfies

$$\tilde{f}_i = \tilde{g} \cdot \tilde{q}_i + \tilde{r}_i \text{ with } \|\tilde{r}_i\| \leq \alpha, i = 1, 2. \tag{6}$$

A near-GCD at accuracy level α will be denoted by α-$GCD(\tilde{f}_1, \tilde{f}_2)$.

Here, the norm of the polynomial means L_1 norm. A near-GCD $\tilde{g}$ is computed via Euclidean Algorithm and a relaxed termination criterion which should be imposed at a given accuracy level α. The algorithm to compute α-GCD of f_1 and f_2 is summarized as follows:

Algorithm 3.1 (α-GCD)
Input: f_1, f_2, α
Output: *α-GCD of f_1 and f_2*
Method:

1. *Compute*

$$f_{j-1} = f_j \cdot q_j + f_{j+1}$$

and

$$s_j^{(i)} = q_j s_{j-1}^{(i)} + s_{j-2}^{(i)}, \ j > i$$

for $j = 2, 3, \cdots$ and $i = 1, 2$ with

$$s_{i-1}^{(i)} = 0, s_i^{(i)} = 1.$$

2. *if $\|s_{j-1}^{(i)} f_{j+1}\| \leq \alpha$, then*

$$f_j = \alpha\text{-}GCD(f_1, f_2).$$

The polynomials f_j generated by the algorithm, f_1 and f_2 are represented as

$$f_i = s_j^{(i)} f_j + s_{j-1}^{(i)} f_{j+1}, \; j > i \geq 1. \tag{7}$$

In our case, f_1 and f_2 should be read as $P(x)$ and $Q(x)$, respectively. Also $s_j^{(1)}$,$s_j^{(2)}$, and f_j should be read as $p(x)$, $q(x)$ and $g(x)$, respectively. Therefore (7) is expressed as

$$\left\{ \begin{array}{lcl} P(x) & = & p(x)g(x) + \delta P(x), \\ Q(x) & = & q(x)g(x) + \delta Q(x), \end{array} \right. \tag{8}$$

where $\delta P(x) = s_{j-1}^{(1)} f_{j+1}$ and $\delta Q(x) = s_{j-1}^{(2)} f_{j+1}$.

4 Accuracy of HRFA

By definition 3.1 and the algorithm 3.1, the hybrid rational function, $p(x)/q(x)$, with the accuracy level α is obtained. We consider, here, the error between $p(x)/q(x)$ and the interpolated rational function, $P(x)/Q(x)$. The following theorem is established.

Theorem 4.1 *Let $P(x)$ and $Q(x)$ be polynomials in $C[x]$ and let $p(x)$ and $q(x)$ be polynomials satisfying $\|\delta P(x)\| \leq \alpha < 1$ and $\|\delta Q(x)\| \leq \alpha < 1$ in (8). Then*

$$\left| \frac{P(z)}{Q(z)} - \frac{p(z)}{q(z)} \right| \leq \left(1 + \left| \frac{P(z)}{Q(z)} \right| \right) \cdot \frac{\alpha}{1-\alpha}, \tag{9}$$

for $z \in [-1,1]$ s.t. $|Q(z)| \geq 1$.

proof

$$\left| \frac{P(z)}{Q(z)} - \frac{p(z)}{q(z)} \right| = \left| \frac{\delta P(z)}{Q(z)} - \frac{P(z)}{Q(z)} \cdot \frac{\delta Q(z)}{Q(z)} \right| \cdot \frac{1}{1 - \frac{\delta Q(z)}{Q(z)}}. \tag{10}$$

The relaxed termination $\|\delta P(z)\| \leq \alpha < 1$ and $\|\delta Q(z)\| \leq \alpha < 1$ implies $\max_{-1 \leq z \leq 1} |\delta P(z)| \leq \alpha < 1$ and $\max_{-1 \leq z \leq 1} |\delta Q(z)| \leq \alpha < 1$ respectively. Thus, rhs of (10) is estimated by (9), for $z \in [-1,1]$ s.t. $|Q(z)| \geq 1$. □

This theorem 4.1 gives the error of HRFA and is the main result of this paper. To show how apparent the error is, we rewrite HRFA algorithm, Algorithm 2.1, as follows:

Algorithm 4.1 (HRFA by near-GCD)
Input: *data set D s.t. $x_i \in [-1,1]$ and parameter α*
Output: *numerical rational interpolation $p(x)/q(x)$*
Method:

1. *Compute rational interpolation $P(x)/Q(x)$ to interpolate D.*

2. *Let* $f_1 = P(x), f_2 = Q(x)$ *input polynomials of algorithm 3.1. Compute* $\alpha - GCD(f_1, f_2)$ *and obtain*

$$p(x)/q(x) = s_j^{(1)}/s_j^{(2)}.$$

After the computation of the procedure *1.* of the algorithm 4.1, accuracy of numerical rational interpolation $p(x)/q(x)$ of the data set D is estimated by Theorem 4.1 as

$$|f(x_i) - p(x_i)/q_(x_i)| \leq (1 + |f(x_i)|) \cdot \frac{c\alpha}{1 - c\alpha}, \tag{11}$$

where positive constant c satisfies the condition $c \cdot |Q(x_i)| \geq 1$ for $i = 0, \cdots, m + n$.

The result (11) shows that our a posteriori error estimate is dominated by the parameter, the accuracy level, α. Similar results may obtained by using other approximate-GCDs but with different accuracy level parameters.

5 An Example of HRFA

We show how the error of HRFA satisfies the error estimate (11). As a practical example, a data set

$$\begin{aligned} D &= \{(-1.0, 0.0384615), (-0.8, 0.588235), (-0.6, 0.1), \\ &\quad (-0.4, 0.2), (-0.2, 0.5), (0, 1.0), (0.2, 0.5), (0.4, 0.2), \\ &\quad (0.6, 0.1), (0.8, 0.0588235), (1.0, 0.00384615)\}, \end{aligned}$$

is considered and approximated by HRFA with the accuracy level $\alpha = 10^{-12}$.

The data set D is obtained by discretization of a function $f(x) = 1/(1 + 25x^2)$. Because of the presence of a roundoff error, rational interpolation of D is obtained by linearized equations such as

$$\begin{aligned} R_{5,5} &= \frac{P(x)}{Q(x)}, \\ P(x) &= -1.58989 \cdot 10^{-14} x^5 + 3.3905 \cdot 10^{-15} x^4 \\ &\quad -21.2844x^3 + 5.53599x^2 + 8.83771x + 1, \\ Q(x) &= -532.11x^5 + 138.4x^4 + 199.658x^3 + 30.536x^2 \\ &\quad +8.83771x + 1. \end{aligned}$$

The leading and the second coefficients of $P(x)$ is smaller than α. Thus deg $(P(x)) = 3$. Since $\min_i |Q(x_i)| = 0.751654$, we put $c = 1.33041$. Thus, the accuracy of symbolic-numeric hybrid rational interpolation is

$$\left| f_i - \frac{p(x_i)}{q(x_i)} \right| \leq 2 \times \frac{1.33041 \cdot 10^{-12}}{1 - 1.33041 \cdot 10^{-12}} = 2.66082 \cdot 10^{-12}.$$

near-GCD(P,Q) with $\alpha = 10^{-12}$ is obtained as

$$10^{-12}\text{-GCD}(P,Q) = -21.2844x^3 + 5.53599x^2 + 8.83771x + 1.00000.$$

Thus rational function is obtained by (8) as

$$r(x) = \frac{p(x)}{q(x)} = \frac{1}{1 + 5.34132 \cdot 10^{-15}x + 25x^2}.$$

The expression can be replaced by

$$r(x) = \frac{p(x)}{q(x)} = \frac{1}{1 + 25x^2}.$$

The result satisfies the theorem 4.1.

6 Summary

In this paper the accuracy of the rational interpolation by HRFA is discussed. A posteriori error estimate is established by using the near-GCD[8]. Similar results may be obtained for other approximate-GCD algorithms. Until now the accuracy of hybrid rational interpolation, i.e., HRFA, is shown only through numerical examples. Our error estimate and the theorem proposed in this paper give the theoretical relation between HRFA and approximate-GCD algorithms. There remains problems on the HRFA such as

1. Our numerical experiments show that the algorithm of HRFA works well for smooth elementary functions $f(x)$ [4]. Under what condition does the close zeros of $P(x)$ and $Q(x)$ occur?
2. How is the HRFA compared to other approximations?

References

[1] D. Braess, *Nonlinear Approximation Theory*, Springer-Verlag, 1986.

[2] J.-P. Berrut and H. D. Mittelmann, Lebesgue Constant Minimizing Linear Rational Interpolation of Continuous Functions over the Interval, Computers Math. Applic., 33, 77–86, 1997.

[3] M. T. Noda and E. Miyahiro, A Hybrid Approach for the Integration of a Rational Function, J. Comput. Appl. Math., 40, 256–268, 1992.

[4] M. T. Noda, E. Miyahiro and H. Kai, Hybrid rational function approximation and its use in the hybrid integration, in *Advances in Computer Methods for Partial Differential Equations VII, IMACS*, 565–571, 1992.

[5] N. Karmarkar and Y. N. Lakshman, Approximate Polynomial Greatest Common Divisors and Nearest Singular Polynomials, ISSAC, 35–39, 1996.

[6] R. M. Corless, P. M. Gianni, B. M. Trager and S. M. Watt, The Singular Value Decomposition for polynomial systems, ISSAC, 195–207, 1995.

[7] A. Schönhage, Quasi-GCD Computations, J. Complexity, 1, 118–137, 1985.

[8] V. Hribernig and H. J. Stetter, Detection and Validation of Clusters of Polynomial Zeros, J. Symb. Comp., 24, 667–681, 1997.

[9] V. Y. Pan, Numerical Computation of A Polynomial GCD and Extensions, preprint(submitted to SIAM Review), 1996.

[10] T. W. Sederberg and G-Z Chang, Best Linear Common Divisors for Approximate Degree Reduction, Computer-aided design, 25, 163–168, 1993.

[11] T. Sasaki and M. T. Noda, Approximate square-free decomposition and root-finding of ill-conditioned algebraic equations, J. Inf. Proc., 12, 159–168, 1989.