Waves in plasma

linear $\times$ nonlinear

Linear waves - small perturbations of a certain state of a system (stationary homogeneous or slowly varying in time and/or space)

Linear expansion of quantities

$$a = a_0 + a_1(\vec{r}, t) \qquad b = b_0 + b_1(\vec{r}, t)$$

a_0, b_0 may be functions of $\vec{r}$, t in general

The products $a_1^2, a_1 \cdot b_1, b_1^2$ are omitted (they are small of the 2nd order)

In spatially unlimited medium $\quad a_1 = \int a_{\vec{k}} \exp(i\vec{k}\vec{r})\, \mathrm{d}\vec{k} \quad$ Fourier expansion

The perturbations evolve independently of each other, it is sufficient to study evolution of periodic perturbations.

We shall be often interested in eigenmodes, i.e. solutions in the form

$$a = \mathrm{Re}\left\{ a_1 \cdot \exp\left[i\left(\vec{k}\vec{r} - \omega t \right) \right] \right\}$$

Eigenmodes are one of the characteristics of a system. We shall search for the dispersion relation $\omega = \omega(\vec{k})$

Way of the system description

- Two-fluid hydrodynamics - simple, but in some cases incomplete description of the system
- Vlasov equation

Classification of waves

- Longitudinal waves **x** transverse waves
- High-frequency (electron) waves x low-frequency waves
- Plasma without stationary *B* x plasma in magnetic field (magnetized plasma)

Plasma waves (Langmuir waves)

(recommended reading – Chen 4.3, 4.4, 7.4 or Nicholson 6.3-6.8,7.3,7.4)

longitudinal waves - velocity $\vec{u} \parallel \vec{k}$

high-frequency (in 1st approximation $m_i \to \infty$)

We assume small deviations from homogeneous stationary state

$$n_e = n_0 + n_1(\vec{r},t) \qquad \vec{u}_e = \underbrace{\vec{u}_0}_{=\vec{0}} + \vec{u}_1(\vec{r},t) \qquad n_0 = Zn_i$$

Continuity equation

$$\frac{\partial n_e}{\partial t} + \mathrm{div}(n_e \vec{u}_e) = 0$$

0. order $\dfrac{\partial n_0}{\partial t} + \mathrm{div}(n_0 \vec{u}_0) = 0$ $\qquad n_0 = const.$

1. order $\dfrac{\partial n_1}{\partial t} + \mathrm{div}\left(n_1 \underbrace{\vec{u}_0}_{0} + n_0 \vec{u}_1 \right) = 0$ $\qquad$ we omit $\cancel{n_1 u_1}$ = 2. order

$$\Rightarrow \qquad \frac{\partial n_1}{\partial t} + n_0\, \mathrm{div}\, \vec{u}_1 = 0$$

Electron density variations —> $\vec{E} = 0 + \vec{E}_1(\vec{r}, t)$

$$\operatorname{div}\vec{E} = \frac{q_e}{\varepsilon_0}(n_e - Zn_i) \qquad\qquad \operatorname{div}\vec{E}_1 = \frac{q_e}{\varepsilon_0} n_1$$

Equation of motion (momentum conservation)

$$\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e \nabla)\vec{u}_e = \frac{q_e}{m_e}\vec{E} - \frac{\nabla p_e}{m_e n_e} - \nu_{ei}(\vec{u}_e - \vec{u}_i)$$

$$\frac{\partial \vec{u}_1}{\partial t} + \nu_{ei} \cdot \vec{u}_1 = \frac{q_e}{m_e}\vec{E}_1 - \frac{\nabla p_1}{m_e n_0} \qquad\qquad (\nabla p_0 = 0)$$

Solution will be assumed in the form $e^{i(\vec{k}\vec{r} - \omega t)}$ (k is real)

$$a(\vec{r}, t) = \mathrm{Re}\left(A \mathrm{e}^{i(\vec{k}\vec{r} - \omega t)}\right) = \mathrm{Re}\left(A^* \mathrm{e}^{-i(\vec{k}\vec{r} - \omega^* t)}\right)$$

$$a = \frac{1}{2}(A \mathrm{e}^{i(\vec{k}\vec{r} - \omega t)} + c.c.)$$

Capital letters – complex amplitudes

Cold plasma without collisions (last term on both sides of eq. motion disappear)

$$\frac{\partial n_1}{\partial t} + n_0 \operatorname{div} \vec{u}_1 = 0$$

$$\operatorname{div} \vec{E}_1 = -\frac{e}{\varepsilon_0} n_1$$

$$\frac{\partial \vec{u}_1}{\partial t} = -\frac{e}{m_e} \vec{E}_1$$

$$\vec{u}, \vec{E} \parallel \vec{k}$$

$$-i\omega N_1 + n_0 i k U_{\parallel} = 0$$

$$i k \tilde{E}_{\parallel} + \frac{e}{\varepsilon_0} N_1 = 0$$

$$-i\omega U_{\parallel} + \frac{e}{m_e} \tilde{E}_{\parallel} = 0$$

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{e^2 n_0}{\varepsilon_0 m_e} n_1 = 0 \qquad \omega_{pe}^2 = \frac{e^2 n_0}{\varepsilon_0 m_e} \qquad U_{\parallel} = \frac{\omega}{k} \cdot \frac{N_1}{n_0} \qquad \tilde{E}_{\parallel} = \frac{i e}{\varepsilon_0 k} N_1$$

Correction when ions are taken into account

$$\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2 \qquad \omega_{pi}^2 = \frac{Z e^2 n_0}{\varepsilon_0 m_i}$$

Reactions on high-frequency field $\vec{E}_1$ (it can be internal or external)

$$\vec{j}_e = -e\left(n_0\vec{u}_1 + \underbrace{n_1\vec{u}_0}_{0} \right) = \underbrace{\frac{ie^2 n_0}{m_e\omega}}_{\sigma_E}\vec{E}_1$$

$$\varepsilon_0 \operatorname{div}\vec{E} = \rho \qquad \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0 \qquad \operatorname{div}\frac{\partial}{\partial t}\left(\varepsilon_0\vec{E}\right) = -\operatorname{div}\vec{j}$$

frequency ω

$$-i\omega\operatorname{div}\left(\varepsilon_0\vec{E} + \frac{i\,\vec{j}}{\omega} \right) = 0$$

$$\operatorname{div}\varepsilon_0\underbrace{\left(1 + \frac{i\,\sigma_E}{\omega\varepsilon_0}\right)}\vec{E} = 0$$

eigenwaves of charge

$$\varepsilon_r = 1 - \frac{e^2 n_0}{\varepsilon_0 m_e \omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\varepsilon\vec{E} = 0$$

$$\vec{E} \neq 0 \quad \Rightarrow \quad \varepsilon_r = 0$$

and thus dispersion relation $\omega = \omega_p$ independent of $k \Rightarrow$ plasma oscillations

Impact of collisions

$$\frac{\partial \vec{u}_1}{\partial t} + \nu_{ei} \cdot \vec{u}_1 = -\frac{e}{m_e} \vec{E}_1 \qquad \frac{\partial^2 n_1}{\partial t^2} + \nu_{ei} \cdot \frac{\partial n_1}{\partial t} + \omega_p^2 n_1 = 0$$

solution $\sim e^{-i\omega t}$ $\omega_{1,2} = -i\frac{\nu_{ei}}{2} \pm \sqrt{\omega_p^2 - \frac{\nu_{ei}^2}{4}}$ $n_1 = n_{10} e^{-i\omega_p t} e^{-\frac{\nu_{ei}}{2}t}$ damped oscil.

Impact of pressure (non-zero temperature)

when $T = 0$ $\vec{\mathrm{v}}_g = \frac{\mathrm{d}\omega}{\mathrm{d}\vec{k}} = \vec{0}$ but when $T \neq 0$ perturbations propagate

spatial shape of the perturbation is preserved, we choose $\vec{k} = k\,\hat{x} \quad \Rightarrow \quad \vec{u}_1 = u_1 \hat{x}$

$$\frac{\partial u_1}{\partial t} = -\frac{e}{m_e} E_1 - \frac{1}{m_e n_0} \underbrace{\frac{\partial}{\partial r_j} P_{1xj}}_{\frac{\partial}{\partial x} P_{1xx}}$$

adiabatic process, $\omega > \nu_{ei} \Rightarrow$ collisions are not able to make the distribution function isotropic

Unperturbed pressure $p_0 = n_0 k_B T_0$ (scalar, T_0 electron temperature)

Pressure perturbation across wavevector is caused only by density perturbation

$$P_{1yy} = P_{1zz} = n_1 k_B T_0 \qquad \left(T_{1\perp} = 0\right)$$

In longitudinal direction, the work by pressure must transform into thermal energy

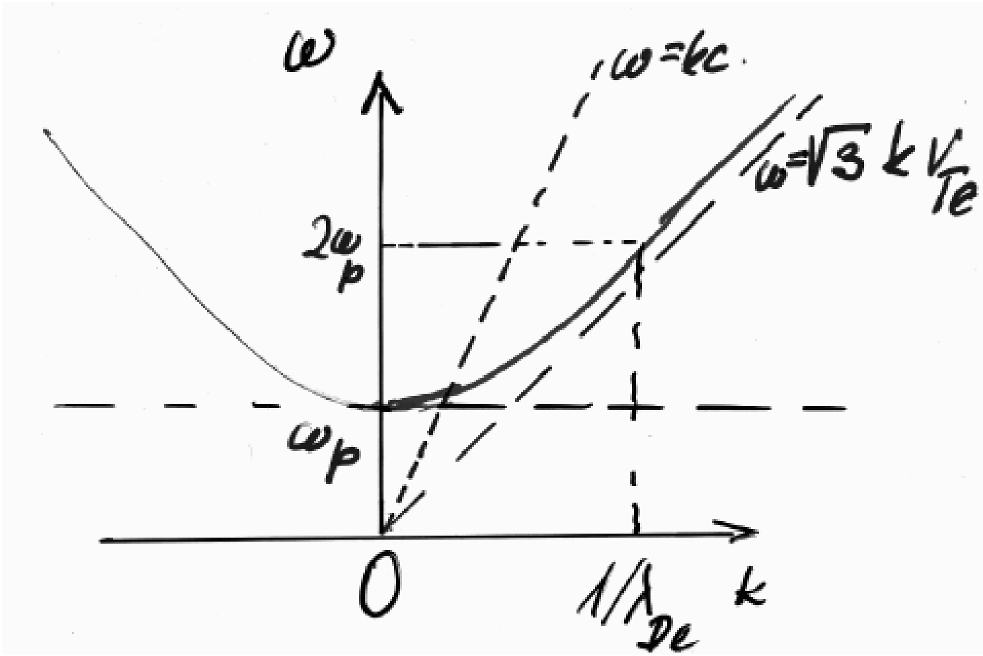
$$\underbrace{\frac{1}{2} n_0 V_0 k_B \,\mathrm{d}T_{\|}}_{\mathrm{d}U} = -p_0 \,\mathrm{d}V = p_0 V_0 \frac{\mathrm{d}n}{n_0} \qquad \mathrm{d}n \to n_1, \quad \mathrm{d}T_{\|} \to T_{1\|}$$

$$\Rightarrow k_B T_{1\|} = \frac{2 p_0}{n_0^2} n_1 = \frac{2 k_B T_0}{n_0} n_1 \qquad P_{1xx} = n_1 k_B T_0 + n_0 k_B T_{1\|} = 3 k_B T_0 n_1$$

In longitudinal direction, electrons are particles with 1 degree of freedom (γ=3)

$$\frac{\partial}{\partial t} u_1 = -\frac{e}{m_e} E_1 - \frac{3 k_B T}{m_e n_0} \frac{\partial n_1}{\partial x} \qquad \Rightarrow \frac{\partial^2 n_1}{\partial t^2} - \frac{3 k_B T_0}{m_e} \frac{\partial^2 n_1}{\partial x^2} + \frac{e^2 n_0}{\varepsilon_0 m_e} n_1 = 0$$

Plasma wave propagates $\quad \omega^2 = \omega_p^2 + 3 k^2 \mathrm{v}_{Te}^2 \quad \left(\mathrm{v}_{Te}^2 = k_B T_e / m_e\right)$



Dispersion relation $\omega^2 = \omega_p^2 + 3k^2 \mathrm{v}_{Te}^2$

$$\mathrm{v}_\varphi = \frac{\omega}{k} = \sqrt{3\mathrm{v}_{Te}^2 + \frac{\omega_p^2}{k^2}}$$

$$\mathrm{v}_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{3k\mathrm{v}_{Te}^2}{\sqrt{\omega_p^2 + 3k^2\mathrm{v}_{Te}^2}}$$

$$\mathrm{v}_g = \frac{3\mathrm{v}_{Te}^2}{\mathrm{v}_\varphi}$$

System with temporal and spatial dispersion $\varepsilon_r^{(l)}\left(\omega,\vec{k}\right) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{3k^2\mathrm{v}_{Te}^2}{\omega^2}$

Description via Vlasov equation

$$\frac{\partial f_e}{\partial t} + \vec{\mathrm{v}}\frac{\partial f_e}{\partial \vec{r}} - e\vec{E}\frac{\partial f_e}{\partial \vec{p}} = 0$$

solution $f_0(\vec{p})$, $\vec{E}_0 = 0$

Perturbations $f_1(\vec{r}, \vec{p}), \vec{E}_1$, $\vec{k} = \hat{x}k$

$$\frac{\partial f_1}{\partial t} + \mathrm{v_x}\frac{\partial f_1}{\partial x} - eE_1\frac{\partial f_0}{\partial p_x} = 0$$

Solution in the form exp(*ikx*-*iωt*)

$$f_1 = i\frac{eE_1}{\omega - k\mathrm{v}_x}\frac{\partial f_0}{\partial p_x}$$

perturbation need not be small for $\mathrm{v_x} = \mathrm{v}_\varphi = \omega / k$

$\Rightarrow$ ***resonance electrons***

$$\mathrm{div}\,\vec{E}_1 = -\frac{e}{\varepsilon_0}n_1 = -\frac{e}{\varepsilon_0}\int f_1 \mathrm{d}\vec{p}$$

$$ikE_1 = -\frac{e}{\varepsilon_0}\int i\frac{eE_1}{\omega - k\mathrm{v}_x}\frac{\partial f_0}{\partial p_x}\mathrm{d}\vec{p}$$

$$ik\varepsilon_0\underbrace{\left(1 + \frac{e^2}{\varepsilon_0 k}\int\frac{1}{\omega - k\mathrm{v}_x}\frac{\partial f_0}{\partial p_x}\mathrm{d}\vec{p}\right)}_{\varepsilon_r}E_1 = 0$$

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\int\frac{g(p_x)}{\left(1 - \frac{k\mathrm{v_x}}{\omega}\right)^2}\mathrm{d}p_x$$

where $g(p_x) = n_0^{-1}\int f_0(\vec{p})\mathrm{d}p_y\mathrm{d}p_z$

When $\mathrm{v}_\varphi = \dfrac{\omega}{k} \gg \mathrm{v}_{Te}$ we use Taylor expansion, resonance electrons are omitted

(for $\mathrm{v}_\varphi > c$ there are no resonance electrons at all)

$$\varepsilon_r \cong 1 - \frac{\omega_p^2}{\omega^2} \int g(p_x) \left(1 + \frac{2k\mathrm{v}_x}{\omega} + \frac{3k^2 \mathrm{v}_x^2}{\omega^2} \right) \mathrm{d}p_x$$

assumed $\langle \mathrm{v}_x \rangle = u_x = 0$

Then $$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2} - \frac{3k^2 \mathrm{v}_{Te}^2}{\omega^2} \frac{\omega_p^2}{\omega^2} \quad \Rightarrow \quad \omega^2 \cong \omega_p^2 + 3k^2 \mathrm{v}_{Te}^2$$

When $\mathrm{v}_\varphi < c$? what to do with pole in integral – answer must be searched via solving initial value problem, i.e. perturbation is given in the initial time t_0 and we follow its evolution

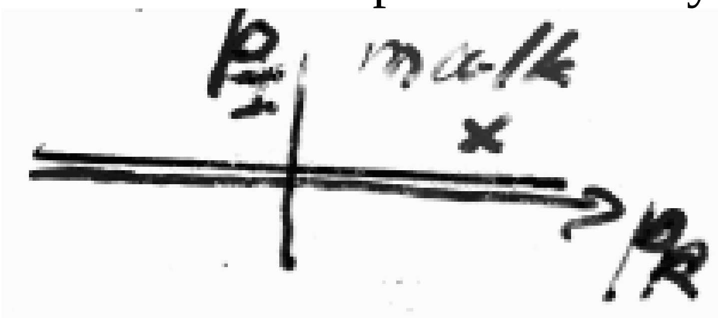
For solving initial value problem, Laplace transform must be applied

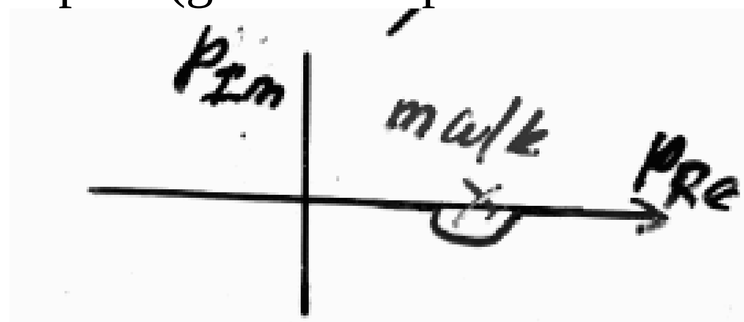
Laplace transform is defined by integral $$A(\omega) = \int_{t_0}^{\infty} a(t)\, \mathrm{e}^{i\omega t} \mathrm{d}t$$ for ω with enough large positive imaginary part (for *a(t)* limited, it is for Im(ω) > 0)

For other ω, Laplace transform is obtained by analytic continuation of function

$$\varepsilon_r = 1 + \frac{m_e \omega_p^2}{k} \int \frac{1}{\omega - k\mathrm{v}_\mathrm{x}} \frac{\mathrm{d}g}{\mathrm{d}p_x}\, \mathrm{d}p_x$$

For Im(ω) > 0 integration path runs below the pole, when doing analytic continuation the path has to stay always below pole (go around pole from below !)





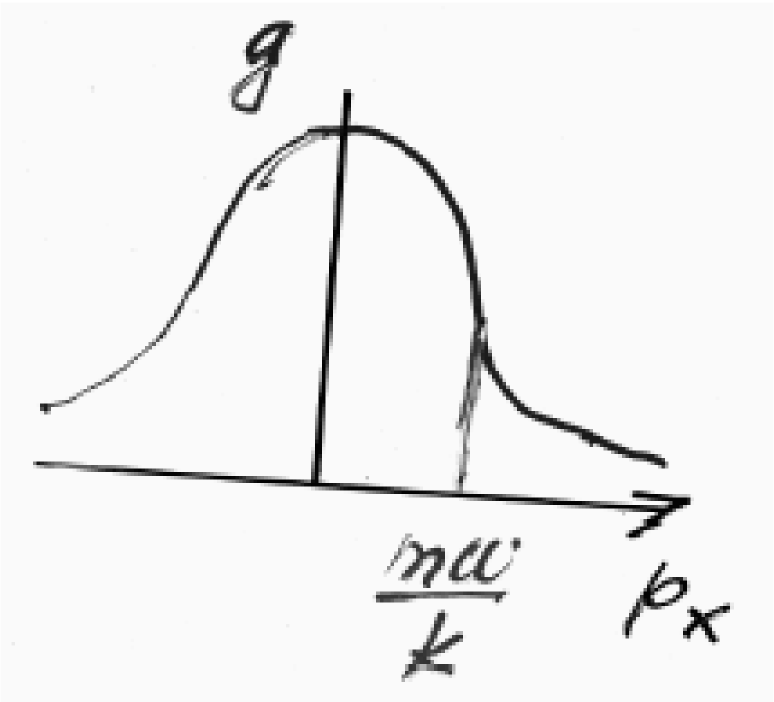
One knows from residue theorem that integral over half-circle is *i*×π×residue

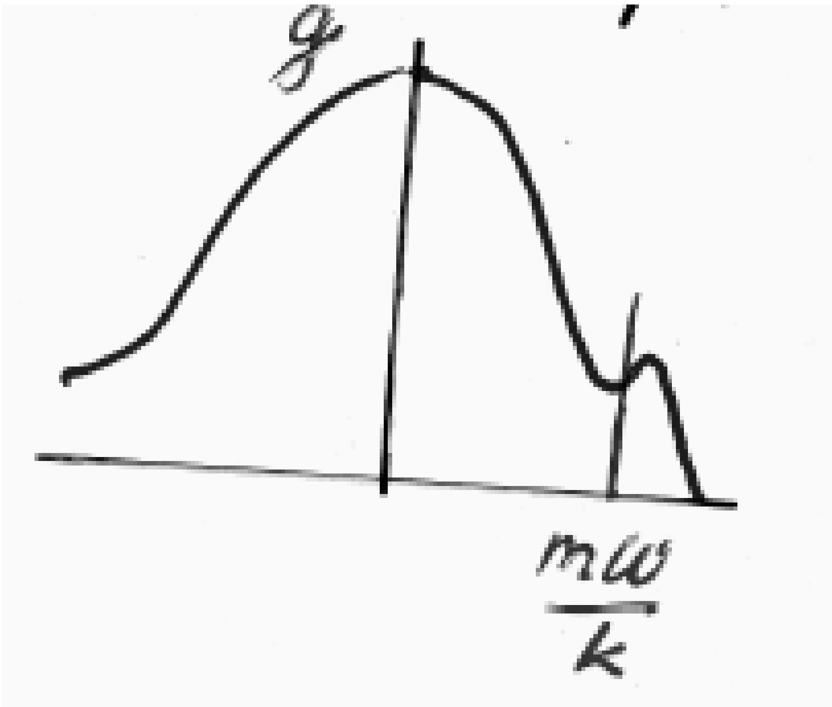
For ω/k << c it is

$$\frac{1}{\omega - k\mathrm{v}_x} = -\frac{m_e}{k} \frac{1}{p_x - \frac{m_e \omega}{k}} = -\frac{m_e}{k} \frac{\mathrm{P}}{p_x - \frac{m_e \omega}{k}} - i\pi \frac{m_e}{k} \delta\left(p_x - \frac{m_e \omega}{k} \right)$$

Here **P** denotes integral in the sense of Cauchy principal value

For ω real it is $$\operatorname{Im} \varepsilon_r(\omega,k) = -\pi\, \omega_p^2 \frac{m_e^2}{k^2} \frac{\mathrm{d}g}{\mathrm{d}p_x}\bigg|_{p_x=\frac{m_e\omega}{k}}$$





$\mathrm{Im}(\varepsilon_r) > 0$ $\mathrm{Im}(\varepsilon_r) < 0$

One searches complex $\omega = \omega_R + i\,\omega_I$ so that $\varepsilon_r(\omega,k) = 0$

Weakly damped (slowly growing) waves $|\omega_I| << \omega_R$

$$\varepsilon_r(\omega_R + i\omega_I) = \mathrm{Re}\,\varepsilon_r(\omega_R) + i\ \mathrm{Im}\,\varepsilon_r(\omega_R) + i\omega_I \frac{\mathrm{d}\,\mathrm{Re}\,\varepsilon_r(\omega_R)}{\mathrm{d}\omega_R} = 0$$

For $\omega_R/k >> \mathrm{v}_{Te}$ it is $$\mathrm{Re}\,\varepsilon_r(\omega_R) = 1 - \frac{\omega_p^2}{\omega_R^2} - \frac{3k^2\mathrm{v}_{Te}^2}{\omega_R^2} = 0$$

and thus $$\omega_R^2 = \omega_p^2 + 3k^2\mathrm{v}_{Te}^2$$

imaginary part of frequency is

$$\omega_I = -\frac{\mathrm{Im}\,\varepsilon_r(\omega_R)}{\dfrac{\mathrm{d}\,\mathrm{Re}\,\varepsilon_r(\omega_R)}{\mathrm{d}\,\omega_R}} = \pi\,\omega_p^2 \frac{m_e^2\omega_R}{2k^2} \left.\frac{\mathrm{d}g}{\mathrm{d}p_x}\right|_{p_x = \frac{m_e\omega_R}{k}}$$

The evolution is $\exp(-i\omega_R t)\exp(\omega_I t)$ - the rate of Landau damping is $\gamma_L = -\omega_I$

For Maxwell's distribution it is $$\omega_I = -\sqrt{\frac{\pi}{8}}\frac{\omega_p^2\omega_R^2}{k^3\mathrm{v}_{Te}^3}\exp\left(-\frac{\omega_R^2}{2\,k^2\mathrm{v}_{Te}^2}\right)$$

Energy of plasma wave

$$-\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{j} \qquad \Rightarrow \qquad \frac{1}{2}\varepsilon_0 \frac{\partial}{\partial t} E^2 = -\vec{j}\vec{E} \qquad\qquad E = \frac{1}{2}\left(\tilde{E}\, e^{-i\omega_R t} + \tilde{E}^*\, e^{i\omega_R t}\right)$$

$\tilde{E}$ is complex amplitude, R denotes real part, we average over time $\langle\ \rangle_{\frac{2\pi}{\omega_R}}$

$$\frac{\varepsilon_0}{4}\frac{\mathrm{d}}{\mathrm{d}\,t}\left|\tilde{E}\right|^2 = -\frac{1}{2}\left(\mathrm{Re}\,\sigma(\omega)\right)\left|\tilde{E}\right|^2 \qquad \mathrm{Re}\,\sigma(\omega) = \mathrm{Re}\,\sigma(\omega_R) - \omega_I \left.\frac{\mathrm{d}\,\mathrm{Im}\,\sigma}{\mathrm{d}\,\omega}\right|_{\omega_R}$$

$$\frac{\varepsilon_0}{4}\frac{\mathrm{d}}{\mathrm{d}\,t}\left|\tilde{E}\right|^2 - \frac{1}{4}\left.\frac{\mathrm{d}\,\mathrm{Im}\,\sigma}{\mathrm{d}\,\omega}\right|_{\omega_R}\frac{\mathrm{d}}{\mathrm{d}\,t}\left|\tilde{E}\right|^2 = -\frac{1}{2}\mathrm{Re}\,\sigma(\omega_R)\left|\tilde{E}\right|^2 \quad \text{used} \quad \frac{\mathrm{d}\,\tilde{E}}{\mathrm{d}\,t} = \omega_I \tilde{E}$$

Conductivity σ related to permittivity ε_r $\quad \varepsilon_r = 1 + \frac{i\sigma}{\omega\varepsilon_0} \rightarrow$ let $\varepsilon_R = \mathrm{Re}(\varepsilon_r)$

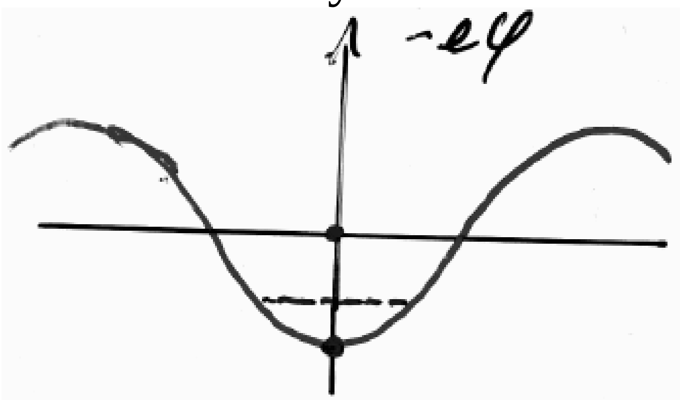
$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Bigg[\underbrace{\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\,\omega}(\omega\varepsilon_0\varepsilon_R)\Big|_{\omega_R}\left|\tilde{E}\right|^2}_{W_{tot}\ =\ \text{energy density}}\Bigg] = -\frac{1}{2}\mathrm{Re}\,\sigma(\omega_R)\left|\tilde{E}\right|^2$$

general expression

(plasma wave $\quad \frac{\mathrm{d}}{\mathrm{d}\,\omega}(\omega\varepsilon_0\varepsilon_R) = 2\varepsilon_0$)

Linear × Non-linear Landau damping

in coordinate system connected to the wave is $\omega_R = 0$



$$E_1 = \tilde{E}\sin kx \quad \text{a} \qquad U_p = -e\varphi = -\frac{e\tilde{E}}{k}\cos kx$$

and electron equation of motion is

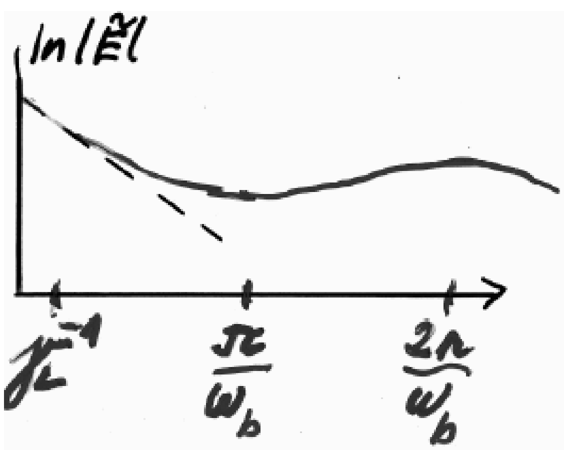
$$m_e\ddot{x} = -e\tilde{E}\sin kx$$

electron oscillates in potential well with frequency

$$\omega_b = \left(\frac{e\,\tilde{E}\,k}{m_e}\right)^{1/2} \qquad \text{(bounce frequency)}$$

for times $t \ll \omega_b^{-1}$ motion is not influenced by field ⇒ Landau damping is linear

for $\gamma_L = -\omega_I > \omega_b$ in time $t = \pi/\omega_b$ electrons start to return energy to wave

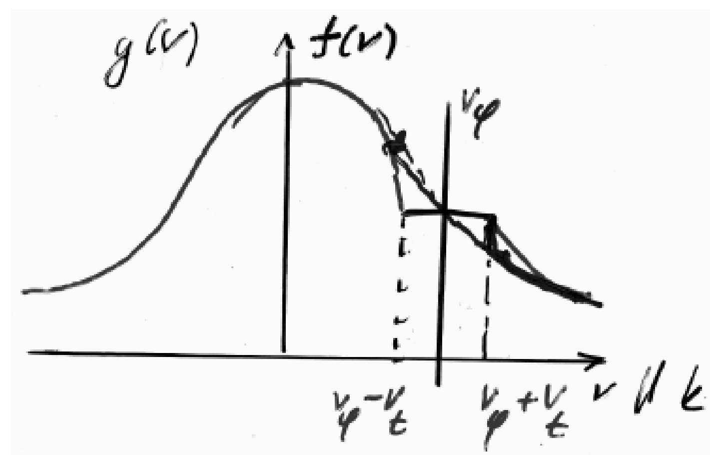


trapped electrons

$$\mathrm{v}_\varphi - \mathrm{v}_t < \mathrm{v} < \mathrm{v}_\varphi + \mathrm{v}_t$$

$$m_e\mathrm{v}_t^2/2 = 2\left|e\varphi_m\right|$$

$$\mathrm{v}_t = 2\left(\frac{e\tilde{E}}{m_e k}\right)^{1/2}$$



BGK modes (Bernstein, Green, Kruskal)

It follows from inhomogeneous equilibrium – accurate **non-linear** solution

Stationary Vlasov equation for particle ***s*** has solution

$$\mathrm{v}_x \frac{\partial f}{\partial x} + q_s E \frac{\partial f}{\partial p} = 0 \qquad f = f\left(\frac{p^2}{2m_s} + q_s \varphi(x)\right) = f(U)$$

Simplest solution for cold untrapped beams

$$n_e(\mathrm{x})\mathrm{v}_e(x) = n_0 \mathrm{v}_{e0} \qquad n_i(\mathrm{x})\mathrm{v}_i(x) = \frac{n_0}{Z}\mathrm{v}_{i0} \qquad \mathrm{v}_e(x) = \sqrt{\mathrm{v}_{e0}^2 + 2e\varphi(x)/m_e}$$

Continuity equation for *e,i* and particle motion in potential field (v_i similarly)

Charge densities of particle are inserted into Poisson equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{e\,n_0}{\varepsilon_0}\left(\frac{\mathrm{v}_{e0}}{\mathrm{v}_e(x)} - \frac{\mathrm{v}_{i0}}{\mathrm{v}_i(x)}\right) = \frac{e\,n_0}{\varepsilon_0}\left\{\left(1 + \frac{2e\varphi}{m_e \mathrm{v}_{e0}^2}\right)^{-1/2} - \left(1 - \frac{2Ze\varphi}{M_i \mathrm{v}_{i0}^2}\right)^{-1/2}\right\}$$

Equation is similar to that for motion in potential field – potential *V(φ)*

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = -\frac{\partial}{\partial\varphi}V(\varphi) \quad \text{where} \quad V(\varphi) = -\frac{n_0}{\varepsilon_0}\left\{m_e \mathrm{v}_{e0}^2\left(1 + \frac{2e\varphi}{m_e \mathrm{v}_{e0}^2}\right)^{1/2} + \frac{M_i \mathrm{v}_{i0}^2}{Z}\left(1 - \frac{2Ze\varphi}{M_i \mathrm{v}_{i0}^2}\right)^{1/2}\right\}$$

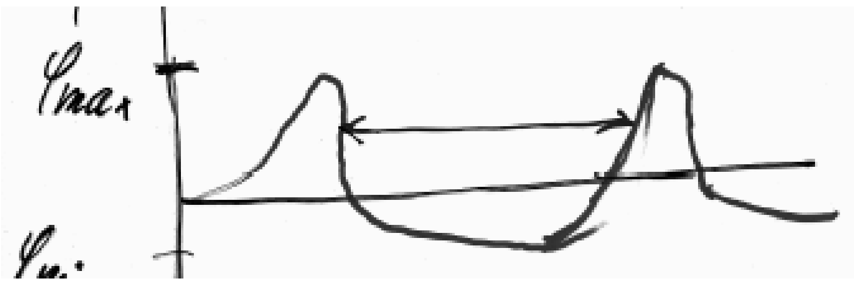
For small φ $\quad |e\varphi(x)| \ll m_e \, \mathrm{v}_{e0}^2 \quad \wedge \quad |e\varphi(x)| \ll \dfrac{M_i \, \mathrm{v}_{i0}^2}{Z}$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}\, x^2} + \frac{n_0 e^2}{\varepsilon_0}\left(\frac{1}{m_e \mathrm{v}_{e0}^2} + \frac{Z}{M_i \mathrm{v}_{i0}^2}\right)\varphi = 0$$

solution

$$\varphi(x) = \varphi_0 \sin\left(x / \lambda_{BGK}\right)$$

$$\lambda_{BGK}^{-2} = \omega_{pe}^2 / \mathrm{v}_{e0}^2 + \omega_{pi}^2 / \mathrm{v}_{i0}^2$$



periodic potential
electrons see it reversely

For any potential, it is possible to construct such stationary distribution of ions and electrons that it creates this given potential

<u>Case-van Kampen modes</u>

One searches for f_1 for given ω, k $\quad f_1 = \tilde{f}_1 \exp\left(ikx - i\omega t\right)$ contain δ function – non-physical

There exist combinations CvK modes that do not contain singularities

High-frequency electrostatic waves in plasma with stationary magnetic field B_0

$\vec{k} \parallel \vec{B}_0$ magnetic field does not influence waves $\Rightarrow$ plasma waves

$\vec{k} \perp \vec{B}_0$ additionally to electrostatic forces, electrons are returned back by magnetic field – cyclotron frequency ω_c

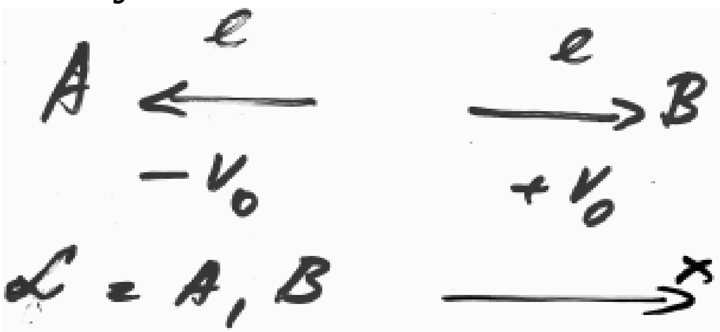
when T=0 $\omega^2 = \omega_p^2 + \omega_c^2 \equiv \omega_h^2$ upper hybrid frequency

upper hybrid waves – plasma waves in direction normal to $\vec{B}_0$
in warm plasma they propagate due to thermokinetic pressure (similarly as plasma waves)

additionally there exist ***linear*** eigenmodes of Vlasov equation that do not have hydrodynamic equivalent – **Bernstein modes**

Stream instabilities (Two-stream instability)

Many situations – motion electrons against ions, motion of electron groups



Simplest situation (*mainly for analytic solution*) – 2 identical electron groups against each other – ions static u_i=0

$n_{A0} = n_{B0} = n_0/2, \qquad Zn_i = n_0$

$v_{Te} << v_0 \qquad E_0 = 0$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(n_\alpha u_\alpha\right) = 0 \qquad \frac{\partial u_\alpha}{\partial t} + (u_\alpha \nabla u_\alpha) = -\frac{eE}{m_e} \qquad \operatorname{div} E = -\frac{e}{\varepsilon_0}\left(n_A + n_B - n_0\right)$$

We solve evolution of linear perturbation $n_{\alpha 1}$, $u_{\alpha 1}$, E_1 ~ exp(ikx-$i\omega t$)

$$-i\omega n_{A1} + ik\left(n_0 u_{A1}/2 - \mathrm{v}_0 n_{A1}\right) = 0 \qquad -i\omega n_{B1} + ik\left(n_0 u_{B1}/2 + \mathrm{v}_0 n_{B1}\right) = 0$$

$$-i\omega u_{A1} - ik\mathrm{v}_0 u_{A1} = -\frac{eE_1}{m_e} \quad -i\omega u_{B1} + ik\mathrm{v}_0 u_{B1} = -\frac{eE_1}{m_e} \quad ikE_1 = -\frac{e}{\varepsilon_0}\left(n_{A1} + n_{B1}\right)$$

Amplitudes of velocities are expressed from equations of motion and we substitute them into continuity equations

$n_{A1} = k\frac{n_0}{2}(-i)\frac{eE_1}{m_e\left(\omega + k\mathrm{v}_0\right)^2}$ $n_{B1} = k\frac{n_0}{2}(-i)\frac{eE_1}{m_e\left(\omega - k\mathrm{v}_0\right)^2}$ and insert them to

Poisson equation $ikE_1 = ik\frac{e^2 n_0}{2\varepsilon_0 m_e}\left(\frac{1}{\left(\omega + k\mathrm{v}_0\right)^2} + \frac{1}{\left(\omega - k\mathrm{v}_0\right)^2}\right)E_1$ and from here

we obtain dispersion relation $1 = \frac{\omega_p^2}{2}\left(\frac{1}{\left(\omega + k\mathrm{v}_0\right)^2} + \frac{1}{\left(\omega - k\mathrm{v}_0\right)^2}\right)$ leading to

$\omega^4 - (2k^2\mathrm{v}_0^2 + \omega_p^2)\,\omega^2 + k^2\mathrm{v}_0^2\left(k^2\mathrm{v}_0^2 - \omega_p^2\right) = 0$, character of the solution depends on the sign of absolute term, if it is > 0, $\omega_1^2 > 0, \omega_2^2 > 0$ then system is stable, if $k^2\mathrm{v}_0^2 < \omega_p^2$, then $\omega_1^2 > 0, \omega_2^2 < 0$ and root with positive imaginary frequency exists – solution grows in time – **instability**

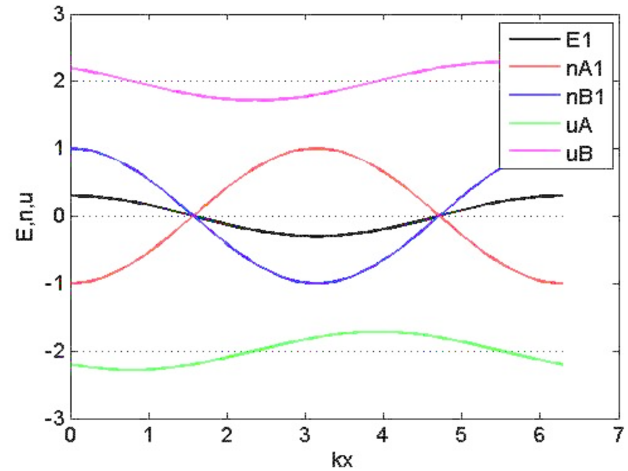
$\omega_{1,2}^2 = k^2\mathrm{v}_0^2 + \frac{\omega_p^2}{2}\left(1 \pm \sqrt{1 + 8\frac{k^2\mathrm{v}_0^2}{\omega_p^2}}\right)$, pro $k^2\mathrm{v}_0^2 < \omega_p^2$ je $\omega_{3,4} = \pm i\sqrt{-\omega_2^2}$

and solution $\omega_3 = i\sqrt{-\omega_2^2}$ is growing exp($-i\omega_3 t$) = exp (γt)

for $k^2 v_0^2 \ll \omega_p^2$ it is $\omega_3 = i\gamma = i|k| v_0$ search for fastest growing mode (k),–

in maximum $$\frac{d(-\omega_2^2)}{d(k^2 v_0^2)} = 0 \quad \Rightarrow \quad k^2 v_0^2 = \frac{3}{8}\omega_p^2; \quad \gamma = \frac{\omega_p}{\sqrt{8}}$$

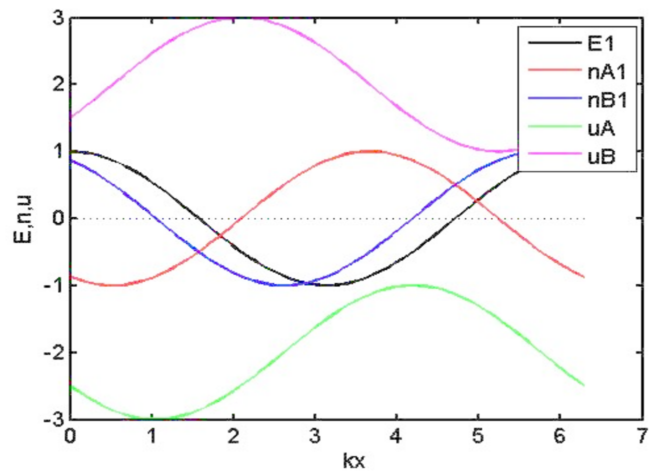
thus fastest growing mode grows only a bit slower than ω_p



How the growing modes look like?

Pro small k for growing mode $\omega = i|k| v_0$ density perturbations of A,B nearly cancel (upper figure – $v_0=2$) Field E_1 is formed only by small sum of densities of order $\sim k^2 v_0^2/\omega_p^2$ growing field $\exp(ikx+kv_0 t)$

Fastest growing mode (lower figure)
One sees nonzero sum of density perturbations of beams A,B
Here special case of growing static perturbation (due to problem symmetry)



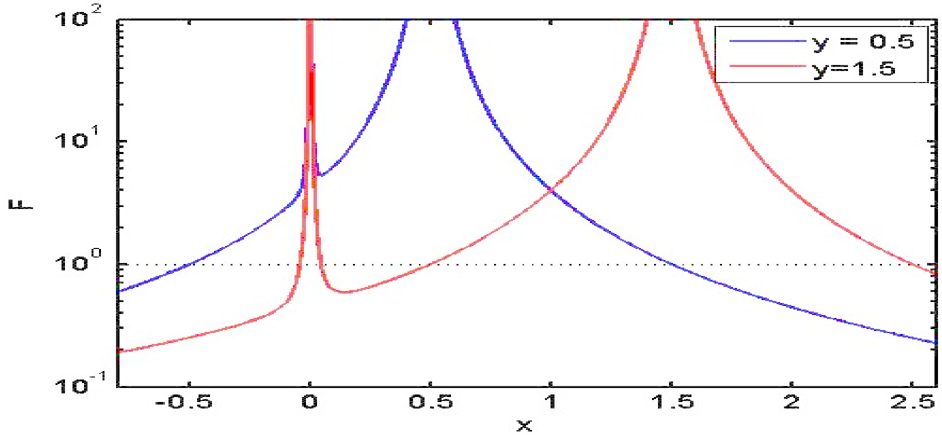
Other case – <u>electron motion against ions</u> with velocity v_0

We introduce $x=\omega/\omega_p$ a $y=kv_0/\omega_p$

Dispersion relation $$1=\frac{m_e/M_i}{x^2}+\frac{1}{(x-y)^2}=F(x,y)$$

for y> boundary, the dispersion relation has 4 real roots – stable system

for y < boundary, the dispersion relation has only 2 real roots – instability



stability boundary $$y^2=\left(1+\sqrt[3]{\frac{M_i}{m_e}}\right)^2\left(\frac{m_e}{M_i}\right)^{2/3}\left(1+\sqrt[3]{\frac{m_e}{M_i}}\right)\approx 1$$ (thus $kv_0\approx\omega_p$)

maximal growth $$\gamma_{\max}=\omega_p\left(\frac{m_e}{M_i}\right)^{1/3}$$