Foundations of kinetic theory – Klimontovich equation

(suggested reading – D.R. Nicholson, Introduction to plasma theory, chap. 3)

phase space $\left(\vec{x},\vec{\mathrm{v}}\right)$ - 6-dimensional space

density of 1 point particle $\vec{x}_1(f),\ \vec{V}_1(t)$ - singular non-zero point

$$N(\vec{x},\vec{\mathrm{v}},t)=\delta\left[\vec{x}-\vec{X}_1(t)\right]\delta\left[\vec{\mathrm{v}}-\vec{\mathrm{V}}_1(t)\right]$$

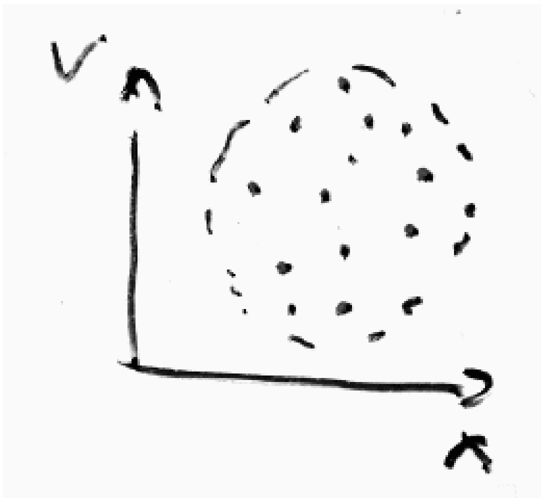
N_{0s} particles of sort „s“ (N_{0s} points)

$$N_s(\vec{x},\vec{\mathrm{v}},t)=\sum_{k=1}^{N_{0s}}\delta\left[\vec{x}-\vec{X}_k(t)\right]\delta\left[\vec{\mathrm{v}}-\vec{\mathrm{V}}_k(t)\right]$$

Equations of motion – for particle number *k*

$$\dot{\vec{X}}_k(t)=\vec{V}_k(t)$$

$$m_s\dot{\vec{V}}_k(t)=q_sE^m\left[X_k(t),t\right]+q_s\vec{V}_k(t)\times\vec{B}^m\left[X_k(t),t\right]$$



Equations for fields – index m = microscopic field

$$\mathrm{div}\,\vec{E}^m = \frac{\rho^m(x,t)}{\varepsilon_0} \qquad \mathrm{rot}\,\vec{E}^m + \frac{\partial \vec{B}^m}{\partial t} = 0$$

$$\mathrm{div}\,\vec{B}^m = 0 \qquad \mathrm{rot}\,\vec{B}^m = \mu_0 \vec{J}^m + \varepsilon_0 \mu_0 \frac{\partial \vec{E}^m}{\partial t}$$

Microscopic charge and current densities

$$\rho^m = \sum_{e,i} q_s \int \mathrm{d}\vec{\mathrm{v}}\, N_s(\vec{x},\vec{\mathrm{v}},t) \qquad \vec{J}^m = \sum_{e,i} q_s \int \mathrm{d}\vec{\mathrm{v}}\,\vec{\mathrm{v}}\, N_s(\vec{x},\vec{\mathrm{v}},t)$$

How evolves N_s in time?

$$N_s = \sum_{k=1}^{N_{0s}} \delta\left[\vec{x} - \vec{X}_k(t)\right] \delta\left[\vec{\mathrm{v}} - \vec{\mathrm{V}}_k(t)\right]$$

$$\frac{\partial N_s}{\partial t} = -\sum_{k=1}^{N_{0s}} \dot{X}_k \nabla_{\vec{x}} \delta\left[\vec{x} - \vec{X}_k(t)\right] \delta\left[\vec{\mathrm{v}} - \vec{\mathrm{V}}_k(t)\right] \quad -\sum_{k=1}^{N_{0s}} \dot{\mathrm{V}}_k \delta\left[\vec{x} - \vec{X}_k(t)\right] \nabla_{\vec{\mathrm{v}}} \delta\left[\vec{\mathrm{v}} - \vec{\mathrm{V}}_k(t)\right]$$

We shall substitute for temporal derivatives of particle position and velocity

$$\frac{\partial N_s(\vec{x},\vec{\mathrm{v}},t)}{\partial t} = -\sum_{k=1}^{N_{0s}} \vec{\mathrm{V}}_k \nabla_x \delta\left[\vec{x}-\vec{X}_k\right]\delta\left[\vec{\mathrm{v}}-\vec{\mathrm{V}}_k\right] -$$

$$-\sum_{k=1}^{N_{0s}} \left\{ \frac{q_s}{m_s} E^m\left[X_k(t),t\right] + \frac{q_s}{m_s}\vec{\mathrm{V}}_k \times \vec{B}^m\left[X_k(t),t\right] \right\} \times \delta\left[\vec{x}-\vec{X}_k\right] \nabla_{\vec{\mathrm{v}}} \delta\left[\vec{\mathrm{v}}-\vec{\mathrm{V}}_k\right]$$

Simple relation $a\delta(a-b) = b\delta(a-b)$ is utilized

$$\frac{\partial N_s}{\partial t} + \vec{\mathrm{v}}\nabla_x N_s + \frac{q_s}{m_s}\left(\vec{E}^m + \vec{\mathrm{v}}\times\vec{B}^m\right)\nabla_{\mathrm{v}} N_s = 0$$

Klimontovich equation *(around year 1960)*

- not usable for plasma description
- used for derivation of suitable equations
- it contains exact trajectories of all particles!

Total derivative along particle trajectory in the phase space – particle density along trajectory does not change

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \left.\frac{dx}{dt}\right|_{\mathrm{traj}} \nabla_x + \left.\frac{d\vec{\mathrm{v}}}{dt}\right|_{\mathrm{traj}} \nabla_{\mathrm{v}} \quad \Rightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t} N_s = 0$$

Fluid interpretation – velocity and force is put inside the derivatives

$$\frac{\partial}{\partial t} N_s + \nabla_x \left(\vec{\mathrm{v}} N_s\right) + \nabla_{\mathrm{v}} \left[\frac{q_s}{m_s} \left(\vec{E}^m + \vec{\mathrm{v}} \times \vec{B}^m \right) N_s \right] = 0$$

analogy in the phase space to the continuity equation $\partial_t \rho + \mathrm{div}(\rho \vec{\mathrm{v}}) = 0$

Averaging over statistical ensemble $f_s\left(\vec{x}, \vec{\mathrm{v}}, t\right) \equiv \left\langle N_s\left(\vec{x}, \vec{\mathrm{v}}, t\right)\right\rangle$

Averaging of fields – one searches for average field that particle see

$\vec{E} \equiv \left\langle \vec{E}^m \right\rangle$ $\qquad$ $\vec{B} \equiv \left\langle \vec{B}^m \right\rangle$

This field may be in general different from field in macroscopic Maxwell´s equations that is given by averaging over space (= problem of acting field – e.g. these fields differ in dielectrics).
In plasmas average acting field = Maxwell´s field!!

We split quantities to average values and deviations from them (fluctuations)

$$N_s = f_s + \delta N_s$$
$$\vec{E}^m = \vec{E} + \delta\vec{E}$$
$$\vec{B}^m = \vec{B} + \delta\vec{B}$$

Averaging of Klimontovich equation

$$\frac{\partial f_s\left(\vec{x},\vec{\mathrm{v}},t\right)}{\partial t} + \vec{\mathrm{v}}\nabla_{\vec{x}} f_s + \frac{q_s}{m_s}\left(\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right)\nabla_{\vec{\mathrm{v}}} f_s = -\frac{q_s}{m_s}\left\langle\left(\delta\vec{E} + \vec{\mathrm{v}}\times\delta\vec{B}\right)\nabla_{\vec{\mathrm{v}}}\delta N_s\right\rangle$$

left side - slowly varying quantities in space, time => collective effects

right side – correlations of fluctuations = collisions

It is basically generalized Boltzmann equations (additionally self-generated fields on the left side, on the right side not only binary correlations, but generalized collision term)

Averaging of Maxwell's equations

$$\left\langle \rho^m \right\rangle \equiv \rho = \sum_{e,i} q_s \int \mathrm{d}\vec{\mathrm{v}}\, f_s(\vec{x},\vec{\mathrm{v}},t) \qquad \left\langle \vec{J}^m \right\rangle \equiv \vec{J} = \sum_{e,i} q_s \int \mathrm{d}\vec{\mathrm{v}}\,\vec{\mathrm{v}}\, f_s(\vec{x},\vec{\mathrm{v}},t)$$

averaging of charge and current densities

after averaging ⇒ **macroscopic** Maxwell's equations

$$\mathrm{div}\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \mathrm{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{rot}\vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

If all correlations are disregarded, collisionless kinetic equation with „self-generated fields“ is obtained

$$\frac{\partial f_s}{\partial t} + \mathrm{v}\nabla f_s + \frac{q_s}{m_s}\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)\nabla_{\mathrm{v}} f_s = 0$$

Vlasov equation

Collisionless description for fast processes in ideal plasmas

Vlasov equation + Maxwell's equations = closed system

We know from the Introduction

$$\frac{\nu_c}{\omega_{pe}} \approx \frac{\ln \Lambda_e}{\Lambda_e} \approx \frac{\ln N_D}{N_D}$$ in ideal plasmas $N_D >> 1$

Hypothetical exercise – particle splitting

$n_0 \to \infty$ $m_e \to 0$ $\Rightarrow \omega_{pe} = konst.$ $e \to 0$

$n_e e = konst.$ $e/m_e = konst.$ $T_e \to 0$ $\Lambda_e \to \infty$ $\lambda_{De} = konst.$

Size of fluctuations $\delta N_s \sim N_0^{1/2} \sim \Lambda_e^{1/2}$

$$\delta E \sim e\delta N_s \sim \frac{1}{N_0} N_0^{1/2} = N_0^{-1/2} \sim \Lambda_e^{-1/2}$$

Kinetic equation (averaged Klimontovich equation)

Left side $f_s \sim N_0 \sim \Lambda_e$

Right side $(\partial f / \partial t)_c = \delta E \delta N_s \approx konst.$

Right side can thus be disregarded for large Λ_e (N_D).

Krook collision term

Sometimes one needs to include collisions at least qualitatively.

For equilibrium (Maxwell) distribution f_M , $(\partial f / \partial t)_c = 0$, any other distribution gradually approaches the equilibrium one via collisions

$$(\partial f / \partial t)_c = -\nu_c (f - f_M)$$

$\Rightarrow$ Krook collision term – the simplest one plausible

It can be generalized for mixtures.