Methods of plasma description

Charged particle motion in external electromagnetic (elmg) fields
Charged particle motion in self-consistent elmg fields
Kinetic equations
Fluid (hydrodynamic) equations
- 2 fluids
- 1 fluid
- magnetohydrodynamics

Particle motion in external fields

(suggested reading– D.R. Nicholson, chap. 2, Chen – chap. 2, 8.4)

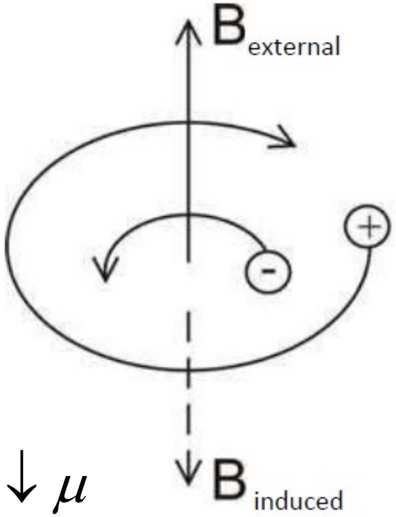
A) Homogeneous fields

a) $\vec{E} = 0$ $\qquad m\frac{d\vec{\mathrm{v}}}{dt} = q\vec{\mathrm{v}} \times \vec{B}$ $\qquad \vec{B} = B\hat{z}$

$$m\dot{\mathrm{v}}_x = qB\mathrm{v}_y \qquad m\dot{\mathrm{v}}_y = -qB\mathrm{v}_x \qquad m\dot{\mathrm{v}}_z = 0$$

$$\ddot{\mathrm{v}}_x = -\left(\frac{qB}{m}\right)^2 \mathrm{v}_x \qquad \ddot{\mathrm{v}}_y = -\left(\frac{qB}{m}\right)^2 \mathrm{v}_y \qquad \omega_c \equiv \frac{|q|B}{m}$$ cyclotron frequency

$$\mathrm{v}_{x,y} = \mathrm{v}_\perp \exp(\pm i\omega_c t + \delta_{x,y}) \qquad \mathrm{v}_x = \mathrm{v}_\perp \exp(i\omega_c t) \qquad \mathrm{v}_y = \frac{m}{qB}\dot{\mathrm{v}}_x = \pm i \mathrm{v}_\perp e^{i\omega_c t}$$



$$x - x_0 = -i\frac{\mathrm{v}_\perp}{\omega_c}e^{i\omega_c t} \qquad y - y_0 = \pm\frac{\mathrm{v}_\perp}{\omega_c}e^{i\omega_c t}$$

Larmor radius

$$r_L = \frac{\mathrm{v}_\perp}{\omega_c} = \frac{m\mathrm{v}_\perp}{|q|B}$$

$\downarrow \mu$

when $\mathrm{v}_\perp = \mathrm{v}_T = \left(\frac{k_B T}{m}\right)^{1/2}$

$$r_L = \frac{(mk_B T)^{1/2}}{|q|B}$$

$$\vec{\mu} = \frac{1}{2}\oint \vec{r}\times\vec{I}\,\mathrm{d}l = -I\vec{S} = \mp\frac{q}{T}S\vec{n} = -\frac{q^2 B}{2\pi m}\pi\left(\frac{m\mathrm{v}_\perp}{qB}\right)^2\vec{n} = -\frac{m\mathrm{v}_\perp^2}{2B}\vec{n}$$

(x_0, y_0, z) gyration center

DIAMAGNETIC MEDIUM - $\mu \sim 1/B \Rightarrow$ not classical magnetics (it has $\mu \sim B$)

b) $E \neq 0$

$$m\frac{d\vec{\mathrm{v}}}{dt} = q(\vec{E} + \vec{\mathrm{v}} \times \vec{B})$$

$$E = (E_x, 0, E_z)$$

$$\frac{d\mathrm{v}_z}{dt} = \frac{q}{m}E_z$$

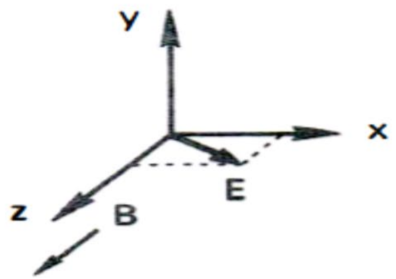
$$\frac{d\mathrm{v}_x}{dt} = \frac{q}{m}E_x \pm \omega_c \mathrm{v}_y$$

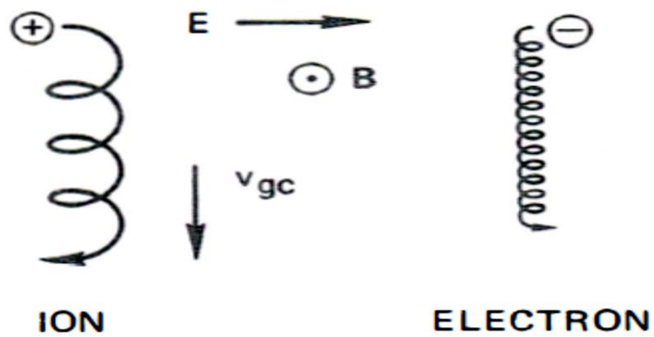
$$\frac{d\mathrm{v}_y}{dt} = \mp\omega_c \mathrm{v}_x$$

$$\ddot{\mathrm{v}}_x = -\omega_c{}^2 \mathrm{v}_x$$

$$\ddot{\mathrm{v}}_y = \pm\omega_c\left(\frac{q}{m}E_x \pm \omega_c \mathrm{v}_y\right) = -\omega_c{}^2\left(\mathrm{v}_y + \frac{E_x}{B}\right)$$

$$\frac{d^2}{dt^2}\left(\mathrm{v}_y + \frac{E_x}{B}\right) = -\omega_c{}^2\left(\mathrm{v}_y + \frac{E_x}{B}\right)$$

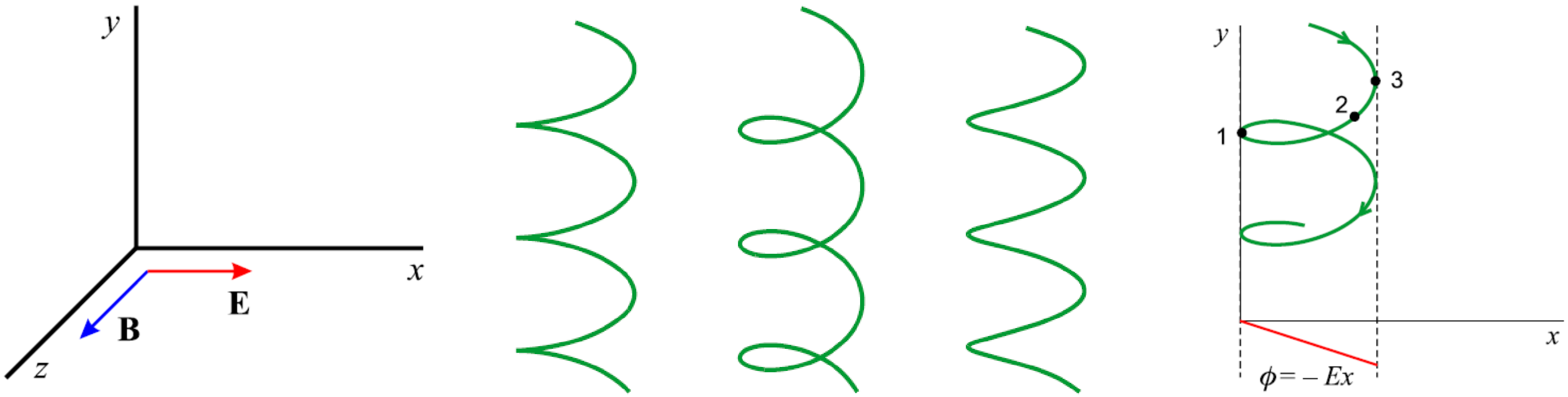




$$\mathrm{v}_x = \mathrm{v}_\perp e^{i(\omega_c t + \delta)}$$

$$\mathrm{v}_y = \pm i \mathrm{v}_\perp e^{i(\omega_c t + \delta)} - \frac{E_x}{B}$$

v_{gc} - station. $\vec{E} + v \times \vec{B} = 0$ gc= gyration center $\vec{v}_{\perp gc} = \dfrac{\vec{E} \times \vec{B}}{B^2} \equiv \vec{v}_E$ drift in E field



here $q > 0$, $v_E = \omega_C r_L$ $v_E < \omega_C r_L$ $v_E > \omega_C r_L$ potential energy - 1 max, 3 min

when v(t=0) = 0, then $v_E = \omega_c r_L = v_\perp$ (cycloid), otherwise trochoid, often case $v_\perp \approx v_T > v_E$

general force

$$\vec{v}_f = \frac{1}{q}\frac{\vec{F} \times \vec{B}}{B^2}$$

e.g. gravity force

$$\vec{v}_g = \frac{m}{q}\frac{\vec{g} \times \vec{B}}{B^2}$$

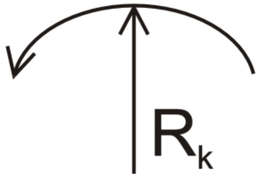
gravity drift

different direction for electrons and ions

$$\vec{j} = n(M + m)\frac{\vec{g} \times \vec{B}}{B^2}$$

gravity current

B) Inhomogeneous $\vec{B}$



$$\vec{F}_{cf} = \frac{m\text{v}_{\|}^{2}}{R_k}\hat{r} = \frac{m\text{v}_{\|}^{2}}{R_k^{2}}\vec{R}_k$$ (centrifugal force)

curvature drift

$$\vec{\text{v}}_R = \frac{1}{q}\frac{\vec{F}_{cf}\times\vec{B}}{B^2} = \frac{m\text{v}_{\|}^{2}}{q}\frac{\vec{R}_k\times\vec{B}}{R_k^{2}B^2}$$

div $\vec{B}$ =0 rot $\vec{B}$ =0 curved field cannot be constant

a) $\nabla|B| \perp \vec{B}$ grad-B drift $B_z = B_z(y) = B_0 + \Delta y\frac{\partial B}{\partial y}$

linear approximation of field during the particle motion on the Larmor circle

$$F_y = -q\text{v}_x B_z(y) = -q\text{v}_\perp(\cos\omega_c t)\left[B_0 \pm r_L(\cos\omega_c t)\frac{\partial B}{\partial y}\right]$$

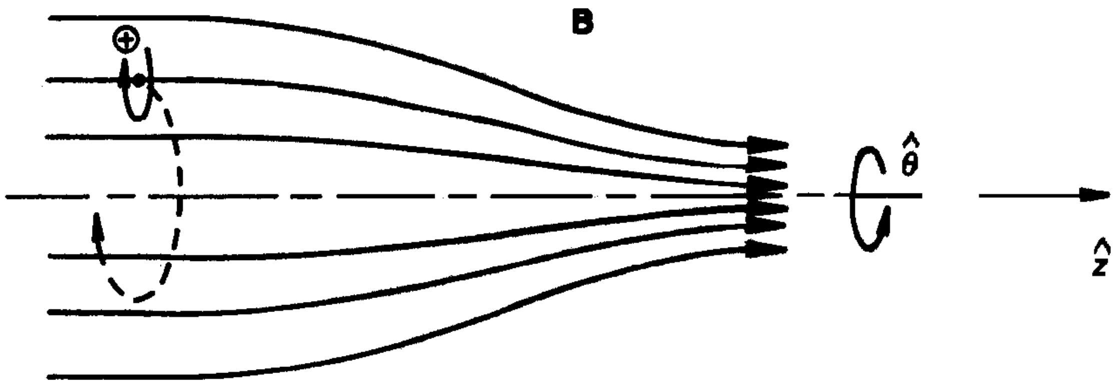
$$\vec{B} = \vec{B}_0 + (\vec{r}\nabla)\vec{B} + \dots \qquad B_z = B_0 + y(\partial B_z/\partial y) + \dots \qquad \langle \cos^2 \omega_c t \rangle = 1/2$$

$$\langle \vec{F}_y \rangle = \pm q \mathrm{v}_\perp r_L \frac{1}{2}\frac{\partial B}{\partial y} = -\frac{m\mathrm{v}_\perp^{\ 2}}{2B}\nabla|B| \qquad \vec{\mathrm{v}}_{\nabla B} = \pm\frac{1}{2}\mathrm{v}_\perp r_L \frac{\vec{B}\times\nabla|\vec{B}|}{B^2}$$

curvature drift and grad-B drift often complement each other

$$\frac{\nabla|B|}{|B|} \cong -\frac{\vec{R}_k}{R_k^{\ 2}} \quad \Rightarrow \quad \vec{\mathrm{v}}_{\nabla B} + \vec{\mathrm{v}}_R = \frac{m}{q}\frac{\vec{R}_k \times \vec{B}}{R_k^{\ 2}B^2}\left(\mathrm{v}_\parallel^{\ 2} + \frac{1}{2}\mathrm{v}_\perp^{\ 2}\right)$$

b) $\nabla|B| \parallel \vec{B}$ Magnetic mirrors



cylindrical symmetry

$$\mathrm{div}\,\vec{B} = 0 \qquad \frac{\partial B_z}{\partial z} \to B_r$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

=> in the neighborhood of axis

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z} \triangleq -\frac{r}{2}\nabla|\vec{B}|$$

$$F_z = q \overbrace{\mathrm{v}_\perp B_r}^{\mathrm{v}_\theta B_r} = -q\mathrm{v}_\perp \frac{r_L}{2} \nabla B = -\underbrace{\frac{m\mathrm{v}_\perp^{\;2}}{2B}}_{\mu} \nabla B \qquad \vec{F} \equiv (\vec{\mu} \cdot \nabla)\vec{B} = -\mu \nabla B$$

μ = magnetic moment

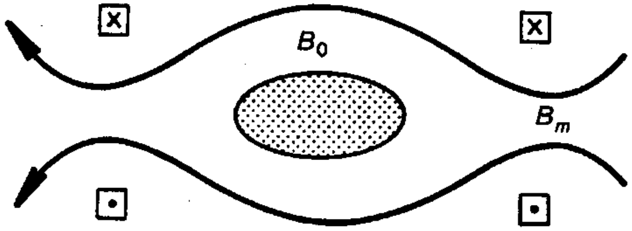
$$\mu ? \;\; \mu = J \cdot S \qquad S = \pi r_L^{\;2} = \pi \frac{m^2}{q^2 B^2} \mathrm{v}_\perp^{\;2} \qquad \mu = \frac{1}{2} \frac{m\mathrm{v}_\perp^{\;2}}{B} \qquad J = \frac{q}{T} = \frac{q\omega_c}{2\pi} = \frac{q^2 B}{2\pi m}$$

Invariance of μ (s…trajectory along the line of force)

$$m\frac{d\mathrm{v}_\parallel}{dt} = -\mu \frac{\partial B}{\partial s} \Rightarrow \frac{d}{dt}\left(\frac{1}{2} m\mathrm{v}_\parallel^{\;2}\right) = -\mu \mathrm{v}_\parallel \frac{\partial B}{\partial s} = -\mu \frac{dB}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2} m\mathrm{v}_\parallel^{\;2} + \frac{1}{2} m\mathrm{v}_\perp^{\;2}\right) = \frac{d}{dt}\left(\frac{1}{2} m\mathrm{v}_\parallel^{\;2} + \mu B\right) = 0 \quad \Rightarrow \quad -\mu \frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

adiabatic invariant



Where is particle with $\vec{v}_0$ from B_0 area reflected?

$$\frac{1}{2} m v_{\perp 0}{}^2 \Big/ B = \frac{1}{2} m v_\perp{}'^2 \Big/ B' \qquad v_\parallel{}' = 0 \qquad v'^2 = v_0{}^2$$

$$\frac{B_0}{B'} = \frac{v_{\perp 0}{}^2}{v_\perp{}'^2} = \frac{v_{\perp 0}{}^2}{v_0{}^2} \equiv \sin^2\theta \qquad \sin^2\theta_m = \frac{B_0}{B_m} = \frac{1}{R_m},$$

where R_m is mirror ratio, it defines loss cone – for $\theta < \theta_m$ particle is not trapped

Adiabatic invariant – quantity that is conserved during slow spatial and temporal variations of the system

Classical mechanics – during periodic motion, action $J = \oint p dq$ is conserved.

Gyration motion $\quad p = m\mathrm{v}_x; \quad q = x$

$$J = \oint m\mathrm{v}_x dx = \int_0^{2\pi/\omega_c} m\mathrm{v}_\perp^{\,2} \sin^2(\omega_c t)\, dt = \frac{\pi m \mathrm{v}_\perp^{\,2}}{\omega_c} = \frac{2m\pi}{q}\mu \Rightarrow \mu = const.$$

When is the adiabatic invariant μ not conserved?

a) cyclotron heating

$\omega \approx \omega_c$, B, E oscillates $\Rightarrow \omega << \omega_c$ does not hold $\Rightarrow \mu \neq$ const.

b) magnetic pumping

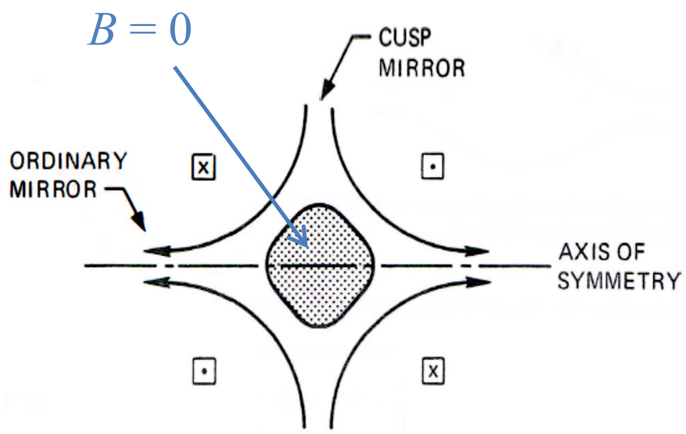
B varies sinusoidally in time, the invariance of μ is broken during particle collisions with moving magnetic mirror

If compression (field increase) occurs during collision, then partly $\mathrm{v}_\perp \to \mathrm{v}_\parallel$

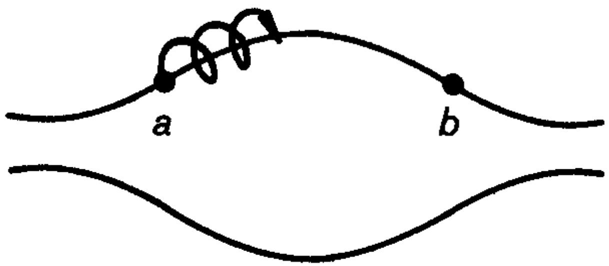
$\rightarrow$ in expansion $\mathrm{v}_\parallel$ is unchanged

c) magnetic cusp in the middle $\quad B = 0 \rightarrow \omega_c = 0$

$\Rightarrow \mu \neq$ const.



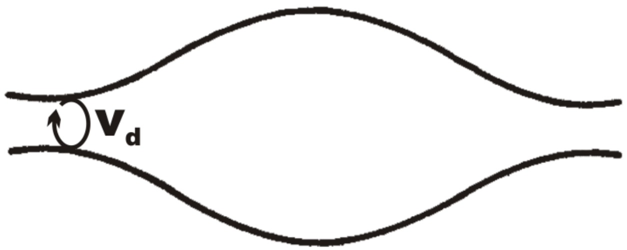
Second adiabatic invariant



a, b …turnabout points

$$J = \int_a^b \mathrm{v}_{\|} ds$$ longitudinal invariant

Third adiabatic invariant



$\vec{\mathrm{v}}_{\nabla B}, \vec{\mathrm{v}}_R \perp \vec{B}, \vec{R}_k$ - drift in direction of angle φ

$J_3 = \oint \mathrm{v}_d dl$ - 3. adiabatic invariant

C) Inhomogeneous E

$E = \hat{x}\cos ky\, E_0$ $\vec{B} = \hat{z}B$

$$m\frac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t} = q\left(\vec{E}(y) + \vec{\mathrm{v}}\times\vec{B}\right) \qquad y = y_0 \pm r_L \cos\omega_c t$$

$$\ddot{\mathrm{v}}_y = 0 = -\omega_c^{\,2}\overline{\mathrm{v}}_y - \omega_c\frac{E_0}{B}\underbrace{\overline{\cos k(y_0 \pm r_L\cos\omega_c t)}}_{\approx\cos ky_0\cdot\left(1-\frac{1}{4}k^2 r_L^{\,2}\right)}$$

$$\vec{\mathrm{v}}_E = \frac{\vec{E}\times\vec{B}}{B^2}\left(1-\frac{1}{4}k^2 r_L^{\,2}\right) = \left(1+\frac{1}{4}r_L^{\,2}\nabla^2\right)\frac{\vec{E}\times\vec{B}}{B^2}$$

<u>Polarization drift (temporally varying E)</u>

$\vec{E}\perp\vec{B}$ $\qquad \dfrac{\partial E}{\partial t}\neq 0$ $\qquad \vec{B} = B_0\hat{z}$ $\qquad \vec{E}(t) = -\dot{E}\,t\,\hat{y}$

$$m\dot{\vec{\mathrm{v}}} = q\vec{E} + q\vec{\mathrm{v}}\times\vec{B}$$

$$\vec{\mathrm{v}} = \vec{\mathrm{v}}_0 + \mathrm{v}_E\hat{x} + \mathrm{v}_p\hat{y}$$ assumption $\mathrm{v}_p = \text{const.}$

$$m\left(\dot{\vec{\mathrm{v}}}_0 + \dot{\vec{\mathrm{v}}}_E\right) = -q\dot{E}t\hat{y} + q\vec{\mathrm{v}}_0\times\vec{B} - q\mathrm{v}_E B_0\hat{y} + q\mathrm{v}_p B_0\hat{x}$$

$$m\dot{\vec{\mathrm{v}}}_0 = q\vec{\mathrm{v}}_0\times\vec{B}$$

cyclotron rotation

$$m\dot{\vec{\mathrm{v}}}_E = q\mathrm{v}_p B_0\hat{x}$$

v_p = polarization drift

$$0 = -q_s\dot{E}\,t\,\hat{y} - q_s\mathrm{v}_E B_0\,\hat{y}$$

v_E – $\vec{E}\times\vec{B}$ drift

$$\mathrm{v}_E\hat{x} = -\frac{\dot{E}t}{B_0}\hat{x} = \frac{\vec{E}\times\vec{B}_0}{B_0^{\;2}} \Rightarrow \dot{\mathrm{v}}_E = -\frac{\dot{E}}{B_0}$$

$$\mathrm{v}_p = -\frac{m\dot{E}}{B_0}\frac{1}{qB_0} = -\frac{m}{qB_0^{\;2}}\dot{E}$$

$$\vec{\mathrm{v}}_p = \frac{m}{qB_0^{\;2}}\frac{d}{dt}\vec{E}$$

$$\vec{J}_p = n_e e(\vec{\mathrm{v}}_{pi} - \vec{\mathrm{v}}_{pe}) = \left(m_e + \frac{M_i}{Z}\right)\frac{n_e}{B_0^2}\frac{d\vec{E}}{dt} = \frac{\rho_M}{B_0^2}\frac{d\vec{E}}{dt}$$

polarization current

PONDEROMOTIVE FORCE

= low frequency force acting on charged particles in inhomogeneous high frequency electromagnetic field

Oscillation energy of charged particles in high frequency field is given by the particle position – so it is some kind of potential energy U and there is force $F = -\nabla U$ that expels charged particles from the area of strong field.

Ponderomotive force acts on any dielectric if its dielectric constant depends on density (electrostriction)!!

Ponderomotive force consists of 2 parts – force caused by particle oscillation in inhomogeneous electric field and force due to magnetic field action on oscillating particle

First, we derive it force caused by particle oscillation in inhomogeneous electric field $\vec{E}$ of frequency ω:

$$\vec{E} = \vec{E}_0(\vec{r})\cos\omega t$$

$$m\ddot{\vec{r}} = q\vec{E} = q\vec{E}_0(\vec{r})\cos\omega t \qquad \overline{\vec{r}} = \vec{r}_0 \qquad \vec{r} = \vec{r}_0 + \vec{r}_1$$

We linearize field variations in oscillation area r_1 and equations of motion are

$$m(\ddot{\vec{r}}_0 + \ddot{\vec{r}}_1) = q\left[\vec{E}_0 + \left(\vec{r}_1\nabla\right)\vec{E}_0\right]\cos\omega t$$

$$m\ddot{\vec{r}}_1 = q\vec{E}_0\cos\omega t \rightarrow \vec{r}_1 = -\frac{q\vec{E}_0}{m\omega^2}\cos\omega t \quad \Rightarrow \ddot{\vec{r}}_0 = \frac{q}{m}\left(\overline{\vec{r}_1\cos\omega t}\,\nabla\right)\vec{E}_0 = -\frac{q^2}{2m^2\omega^2}\left(\vec{E}_0\nabla\right)\vec{E}_0$$

So low frequency force is $\vec{F}_E = -\dfrac{q^2}{2m\,\omega^2}\left(\vec{E}_0\nabla\right)\vec{E}_0$

High frequency magnetic field acts on oscillating particle

$$\vec{B} = \vec{B}_0(\vec{r})\sin\omega t \qquad \vec{B}_0 = -\frac{1}{\omega}\operatorname{curl}\vec{E}_0 \qquad \vec{\mathrm{v}} = \frac{q}{m\,\omega}\vec{E}_0\sin\omega t$$

The force is then given by the following expression

$$\vec{F}_B = \overline{q\vec{\mathrm{v}}\times\vec{B}} = \frac{q^2}{m\,\omega}\vec{E}_0\times\vec{B}_0\overline{\sin^2\omega t} = -\frac{1}{2}\frac{q^2}{m\,\omega^2}\vec{E}_0\times\operatorname{rot}\vec{E}_0$$

The total ponderomotive force is then given by the sum of the forces

$$\vec{F}_p = \vec{F}_E + \vec{F}_B = -\frac{1}{2}\frac{q^2}{m\omega^2}\left[\left(\vec{E}_0\nabla\right)\vec{E}_0 + \vec{E}_0 \times \mathrm{curl}\,\vec{E}_0\right] = -\frac{1}{4}\frac{q^2}{m\omega^2}\nabla E_0^2 = -\frac{1}{4}\frac{q^2}{m\omega^2}\nabla\left|E_0\right|^2$$

We have used a real amplitude during the derivation, however, the field may be phase shifted in general, thus, the general expression includes absolute value of the field amplitude.

Low frequency force acting on the particle thus reads, as follows

$$\vec{F}_p = -\frac{q^2}{4m\omega^2}\nabla\left|E_0\right|^2 \qquad \vec{F}_p = -\nabla W_{osc}$$ force equal to -gradient of potential energy

$$W_{osc} = \frac{1}{2}m\overline{\mathrm{v}^2} = \frac{1}{2}m\frac{q^2E_0^{\ 2}}{m^2\omega^2}\overline{\cos^2\omega t} = \frac{1}{4}\frac{q^2}{m\omega^2}E_0^{\ 2}$$

There exists also high frequency force of frequency $2\omega_0$.

For field containing 2 frequencies there ∃ forces with sum and difference of ω.